Two Geometric Representations of Confidence Intervals for Ratios of Linear Combinations of Regression Parameters: An Application to the NAIRU.

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Although the Fieller Method for the construction of confidence intervals for ratios of normally distributed random variables has been shown to be a superior method to the delta method it is infrequently used. We feel that researchers do not have an intuition as to how the Fieller Method operates and how to interpret the non-finite intervals that it may produce. In this note we present two simple geometric representations of the Fieller interval and demonstrate how they can be used to interpret the estimation of the NAIRU.

Key words: Fieller Method, confidence ellipse, 1st derivative function

JEL: C12, C20, E24

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I. Introduction

Drawing inferences from the ratio of regression coefficients is elemental to a number of econometric applications. Generally, the results of Monte Carlo simulations to compare the Fieller Method (Fieller 1932, 1954) with other procedures for the construction of confidence intervals indicates that the Fieller Method consistently out-performs the widely used Delta method and is comparable and in some cases, superior to the more complex Bayesian and bootstrap techniques. (See Hirschberg and Lye 2004).

Dufour (1997) proposed that ratios of regression parameter problems be subject to confidence intervals based on the Fieller type methods. Its application can be found for: long-run elasticities in dynamic energy demand models (Bernard et al. 2005); mean elasticities obtained from linear regression models (Valentine 1979); non-accelerating inflation rate of unemployment, the NAIRU (Staiger et al. 1997); steady state coefficients in models with lagged dependent variables (Blomqvist 1973) and the extremum of a quadratic model (Hirschberg and Lye 2004).

We propose that the non-intuitive way in which the Fieller Method is traditionally presented as the solution to a quadratic equation is partly to blame for its infrequent use. In this note we present two geometric representations of the Fieller Method which may lead to an enhanced intuition for these confidence intervals. Both of these approaches can be implemented using existing econometric software (see Hirschberg and Lye 2007).

II. The Fieller Method

The Fieller Method (Fieller 1932, 1954) provides a general procedure for constructing confidence limits for statistics defined as ratios. Zerbe (1978) defines a version of Fieller's Method in the regression context, consider the ratio $\psi = \frac{\rho}{\phi}$ where

$\rho = \mathbf{K}'\beta$ and $\phi = \mathbf{L}'\beta$ are linear combinations of the parameters from the regression, $\mathbf{Y}_{T\times 1} = \mathbf{X}_{T\times k}\beta_{k\times 1} + \varepsilon_{T\times 1}$, $\varepsilon \sim (\mathbf{0}_{T\times 1}, \sigma^2\mathbf{I}_{T\times T})$. The OLS estimators are defined as $\hat{\beta} = (\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}'\mathbf{Y}$, $\hat{\sigma}^2 = \hat{\varepsilon}'\hat{\varepsilon}/(T-k)$, and the vectors $\mathbf{K}_{k\times 1}$ and $\mathbf{L}_{k\times 1}$ are known constants. Under the usual assumptions, the parameter estimates are asymptotically normally distributed according to $\hat{\beta} \sim N\left(\beta, \sigma^2\left(X'X\right)^{-1}\right)$. The ratio ψ is estimated as $\hat{\psi} = \frac{\hat{\rho}}{\hat{\phi}}$ where $\hat{\rho} = \mathbf{K}'\hat{\beta}$ and $\hat{\phi} = \mathbf{L}'\hat{\beta}$.

The Fieller $100(1-\alpha)$% confidence interval for ψ is determined by solving the quadratic equation $a\psi^2 + b\psi + c = 0$, where $a = (\mathbf{L}'\hat{\beta})^2 - t^2_{\alpha/2}\mathbf{L}'(\mathbf{X}'\mathbf{X})^{-1}\mathbf{L}\hat{\sigma}^2$, $b = 2\left[t^2_{\alpha/2}\mathbf{K}'(\mathbf{X}'\mathbf{X})^{-1}\mathbf{L}\hat{\sigma}^2 - (\mathbf{K}'\hat{\beta})(\mathbf{L}'\hat{\beta})\right]$ and $c = (\mathbf{K}'\hat{\beta})^2 - t^2_{\alpha/2}\mathbf{K}'(\mathbf{X}'\mathbf{X})^{-1}\mathbf{K}\hat{\sigma}^2$.

When $a > 0$, the two roots of the quadratic equation, $\left(\psi_1, \psi_2\right) = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$, define finite confidence bounds. The condition $a > 0$, is true when the hypothesis test $H_0 : \mathbf{L}'\beta = 0$ is rejected at the a level of significance (Buonaccorsi 1979). Alternatively, if $H_0 : \mathbf{L}'\beta = 0$ cannot be rejected the resulting confidence interval may be the complement of a finite interval (when $b^2 - 4ac > 0$, $a < 0$) or of the whole real line (when $b^2 - 4ac < 0$, $a < 0$). These conditions are discussed in Scheffé (1970) and Zerbe (1982).

III. Confidence Bounds of the Linear Combination (CBLC)

The $100(1-a)$% confidence interval for $g = \left\{\left(\mathbf{K}'\hat{\beta}\right) - \left(\mathbf{L}'\hat{\beta}\right)\psi\right\}$ given by:

$$\left\{\left(\mathbf{K}'\hat{\beta}\right) - \left(\mathbf{L}'\hat{\beta}\right)\psi\right\} \pm t_{\alpha/2}\sqrt{\left\{\hat{\sigma}^2\mathbf{K}'(\mathbf{X}'\mathbf{X})^{-1}\mathbf{K} - 2\left(\hat{\sigma}^2\mathbf{K}'(\mathbf{X}'\mathbf{X})^{-1}\mathbf{L}\right)\psi + \left(\hat{\sigma}^2\mathbf{L}'(\mathbf{X}'\mathbf{X})^{-1}\mathbf{L}\right)\psi^2\right\}} \quad (1)$$

where $t_{\alpha/2}$ is the value from the t distribution with an $(\alpha/2)\%$ level of significance and $T-k$ degrees of freedom.

The ratio $(\hat{\psi})$ should satisfy $(\mathbf{K}'\hat{\beta})-(\mathbf{L}'\hat{\beta})\hat{\psi}=0$ and the $100(1-a)\%$ bounds for $\hat{\psi}$ are found by solving:

$$\begin{aligned}&\left\{(\mathbf{K}'\hat{\beta})-(\mathbf{L}'\hat{\beta})\psi\right\}^2\\&\quad -t_{\alpha/2}^2\left\{\hat{\sigma}^2\mathbf{K}'(\mathbf{X}'\mathbf{X})^{-1}\mathbf{K}-2\left(\hat{\sigma}^2\mathbf{K}'(\mathbf{X}'\mathbf{X})^{-1}\mathbf{L}\right)\psi+\left(\hat{\sigma}^2\mathbf{L}'(\mathbf{X}'\mathbf{X})^{-1}\mathbf{L}\right)\psi^2\right\}=0\end{aligned} \tag{2}$$

This expression (2) can be written as $a\psi^2+b\psi+c=0$, where a, b and c are defined as in the Fieller Method described in Section II.

This result implies a geometric representation of the Fieller-type confidence interval can be implemented using any statistics software that can predict a linear function of the estimated coefficients of a regression and with a confidence interval (see Hirschberg and Lye 2007).

IV. Confidence Ellipse (CE) Geometric Representation

We can define the confidence ellipse for two regression parameters or two linear combinations of regression parameters such as ρ and ϕ. A regression of the form $\mathbf{Y}_{T\times 1}=\mathbf{X}_{T\times k}\beta_{k\times 1}+\varepsilon_{T\times 1}$ can always be transformed to another regression of the form:

$\mathbf{Y}_{T\times 1}=(\mathbf{Z_1})_{T\times 2}\gamma_{2\times 1}+(\mathbf{Z_2})_{T\times(k-2)}\theta_{(k-2)\times 1}+\varepsilon_{T\times 1}$ where $\gamma=[\rho \quad \phi]'$, θ a k -2 vector of parameters, $\mathbf{R}_{2\times k}=[\mathbf{K}' \quad \mathbf{L}']'$, $\mathbf{Z_1}=\mathbf{XR}^+$, $\mathbf{R}^+$ is the generalized inverse of $\mathbf{R}$, $\mathbf{Z_2}=\mathbf{XA}$, and $\mathbf{A}$ is the matrix of $k-2$ eigenvectors corresponding to the zero valued eigenvalues of $\mathbf{R}'\mathbf{R}$ (see Hirschberg, Lye and Slottje (2005)). The marginal $100(1-\alpha)\%$ ellipse for a combination of the parameters in γ is:

$$(\gamma-\gamma)'\,\hat{\sigma}^{-2}\left(\mathbf{Z}_1'\mathbf{M}_2\mathbf{Z}_1\right)(\gamma-\hat{\gamma}) \le F_\alpha\left(1,T-k\right) \tag{3}$$

where $\mathbf{M}_2 = \mathbf{I} - \mathbf{Z}_2\left(\mathbf{Z}_2'\mathbf{Z}_2\right)^{-1}\mathbf{Z}_2'$.

The solution to the constrained optimization problem defined as:

$$\mathrm{L} = \psi - \lambda\left(\begin{bmatrix}(\hat{\rho}-\psi\phi) & (\hat{\phi}-\phi)\end{bmatrix}\begin{bmatrix}\omega_{11} & \omega_{12}\\ \omega_{12} & \omega_{22}\end{bmatrix}\begin{bmatrix}(\hat{\rho}-\psi\phi)\\ (\hat{\phi}-\phi)\end{bmatrix} - F_\alpha\left(1,T-k\right)\right) \tag{4}$$

where ω_{ij} are elements of $\mathbf{\Omega} = \hat{\sigma}^{-2}\left(\mathbf{Z_1 M_2 Z_1}\right)$ and λ is the Lagrange multiplier, has two roots that are equivalent to the Fieller interval. (Hirschberg and Lye 2007).

(Figure1)

The solution to this constrained optimization problem can be found using a diagram of the ellipse defined by (3). Following von Luxburg and Franz (2004), the ratio $\hat{\psi} = \dfrac{\hat{\rho}}{\hat{\phi}}$ is the slope of a ray from the origin (0,0) through the point ($\hat{\rho},\hat{\phi}$). If (0,0) is not within the ellipse, two more rays from the origin can be constructed that are tangent to the ellipse. If $H_0 : \phi = 0$ is rejected at the $\alpha\%$ level of significance (see Figure 1) the ellipse does not cut the y-axis and a finite confidence bound can be defined where the tangent rays intersect the line defined by $\phi = 1$. In Figure 2 we show the case when $H_0 : \phi = 0$ cannot be rejected and the ellipse cuts the y-axis there is one finite bound, here the ratio has a lower bound but no upper bound. When (0,0) is within the ellipse the interval is then the whole real line.

(Figure2)

The confidence ellipse produced in Eviews 6.0 is specified as the joint ellipse $(\gamma-\gamma)'\,\hat{\sigma}^{-2}\left(\mathbf{Z}_1'\mathbf{M}_2\mathbf{Z}_1\right)(\gamma-\hat{\gamma}) = 2F_\alpha\left(2,T-k\right)$. To obtain the marginal confidence ellipse (3) we define $\tilde{\alpha}$ such that $F_\alpha\left(1,T-k\right) = 2F_{\tilde{\alpha}}\left(2,T-k\right)$. In Stata 8 as $T-k$ is large, the 95% confidence ellipse can be obtained using the program *ellip* (Alexandersson 2004) by specifying the boundary constant using *chi2* with 1 degree of freedom, (see Hirschberg and Lye 2007).

V. An application to the Non-Accelerating Inflation Rate of Unemployment (NAIRU)

Following the estimation proposed by Gruen et al. (1999) we estimate:

$$\begin{aligned}\Delta_4 \ln ULC_t - \Delta_4 \ln P_{t-1} &= \alpha_1\left(\Delta_4 \ln P_t^* - \Delta_4 \ln P_{t-1}\right) + \alpha_2 U_t + \alpha_3 \Delta U_{t-1} \\ &+ \alpha_4\left(\Delta_4 \ln ULC_{t-1} - \Delta_4 \ln P_{t-2}\right) + \alpha_5\left(\Delta_4 \ln ULC_{t-1} - \Delta_4 \ln ULC_{t-4}\right) + \alpha_6 + \varepsilon\end{aligned} \tag{5}$$

Where ULC = unit labour costs per person, and is equal to wages per person divided by non-farm productivity per person; P = CPI, P^* = expected price level; U = rate of unemployment; Δ = 1 period change; and Δ_4 = 4 period change. An estimate of the NAIRU is defined as $\hat{U}^* = \hat{a}_6 / -\hat{a}_2$, where $\hat{a}_6,\ \hat{a}_2$ are the OLS estimates from (5).

(Table1)

Table 1 presents the estimates of (5) using quarterly Australian data from Lye and McDonald (2006) for the period 1985:1 – 2003:4. Based on these estimates $\hat{U}^* = {}^{1.328}\!/_{0.246} = 5.40\%$ and the estimated 95% confidence interval for $\hat{U}^*$ based on the Delta method is given by as [3.120%, 7.682%].

(Figure3)

To obtain the 95% Fieller confidence bounds using the CBLE approach, in Figure 3, we plot $g = \hat{a}_6 + \hat{a}_2 U^*$ with the 95% confidence bounds of LY given by,

$$\left(\hat{a}_6 + \hat{a}_2 U^*\right) \pm t_{\alpha/2}\sqrt{\hat{\sigma}^2_{\hat{a}_6} + 2\sigma_{\hat{a}_2\hat{a}_6} U^* + \hat{\sigma}^2_{\hat{a}_2}\left(U^*\right)^2} \tag{6}$$

The Fieller confidence bounds are defined as the points where y = 0. From Figure 3, the 95% Fieller Interval is [-10.11%, 6.91%].

(Figure4)

In Figure 4 we provide the Fieller interval using the CE by extending two rays from the origin that are tangent to the ellipse. The values of the upper and lower limits of the Fieller interval in this case are finite and are defined at the points where the rays from the origin cut the line defined by $x = 1$.

VI. Conclusions

In this note we demonstrate two geometric representations of the Fieller confidence interval. From these geometric representations one can see how the distribution of the estimates of the two variables influences the nature of the confidence interval. Specifically, these methods demonstrate how the Fieller Method may not result in two finite bounds.

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Figure 1: An Example of Finite Confidence Bounds

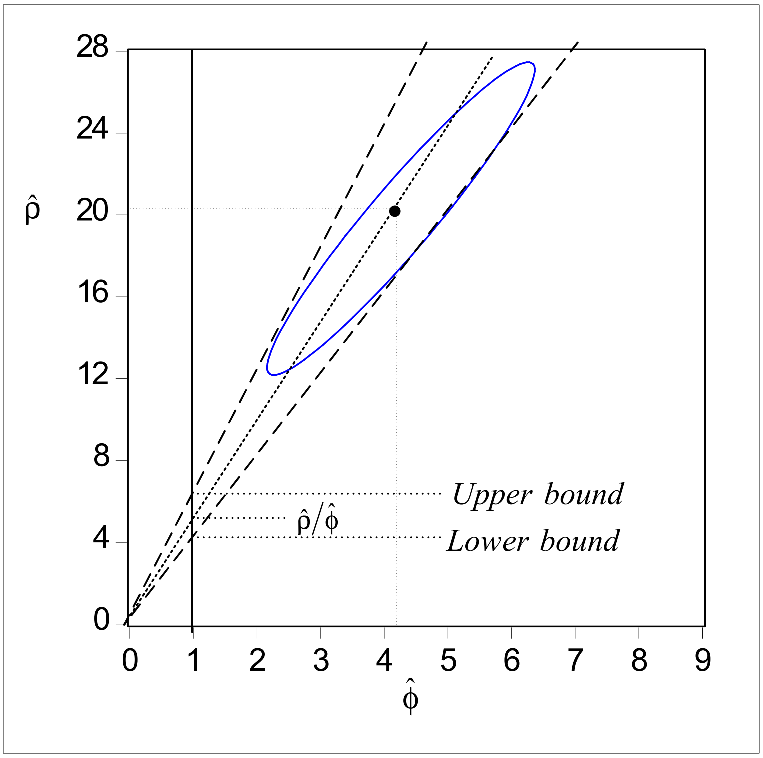

Figure 2: An Example of a Complement of a Finite Interval

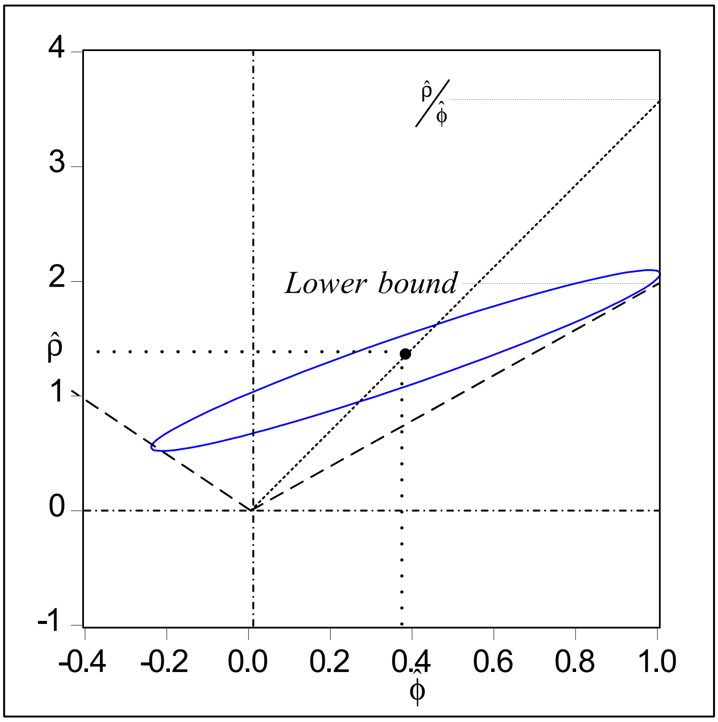

Table 1: Phillips Curve Estimates for Australia 1985:1 – 2003:4

Parameter	**Estimate**	**Standard Error**	***p*-value**
$\hat{a}_1$	0.16716	0.07790	0.0354
$\hat{a}_2$	-0.24589	0.11118	0.0303
$\hat{a}_3$	-0.28008	0.47844	0.5602
$\hat{a}_4$	0.58431	0.07351	0.0000
$\hat{a}_5$	0.55623	0.10064	0.0000
$\hat{a}_6$	1.32780	0.83375	0.1157
$\hat{\sigma}_{\hat{a}_2\hat{a}_6} = -0.090$	R^2=0.693	Number of observations = 76	

Figure 3: The Fieller Confidence bounds for the NAIRU using Confidence Bounds Approach

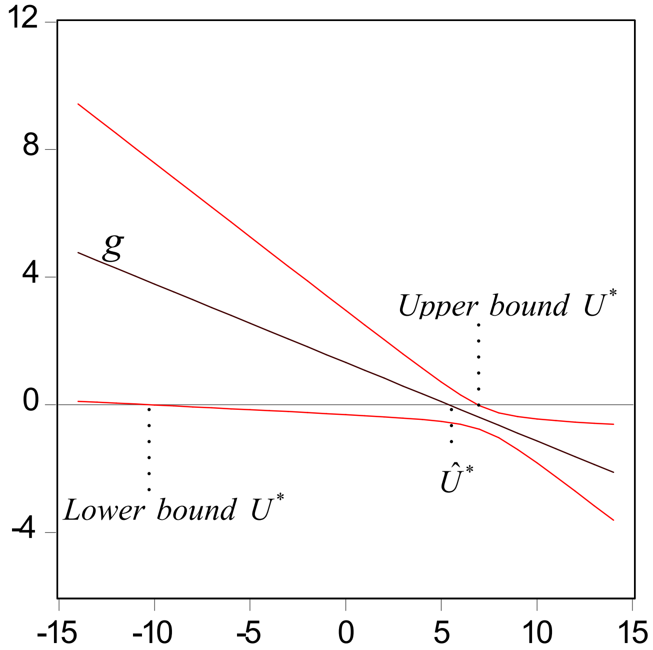

Figure 4: The Fieller Confidence bounds for the NAIRU using Confidence Ellipse Approach