Inferential Methods for Elasticity Estimates[1]

J. G. Hirschberg[2], J. N. Lye[3] and D. J. Slottje[4]

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Abstract

Elasticities are often estimated from the results of demand analysis however, drawing inferences from them may involve assumptions that could influence the outcome. In this paper we investigate one of the most common forms of elasticity which is defined as a ratio of estimated relationships and demonstrate how the Fieller method for the construction of confidence intervals can be used to draw inferences.

We estimate the elasticities of expenditure from Engel curves using a variety of estimation models. Parametric Engel curves are modelled using OLS, MM robust regression, and Tobit. Semiparametric Engel curves are estimated using a penalized spline regression. We demonstrate the construction of confidence intervals of the expenditure elasticities for a series of expenditure levels as well as the estimated cumulative density function for the elasticity evaluated for a particular household.

Key words: Engel curves, Fieller method, Tobit, robust regression, semiparametric

JEL codes: C12, C13, C14, C24, D12

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[2] Department of Economics, University of Melbourne, Melbourne, Vic 3010, Australia.
[3] Department of Economics, University of Melbourne, Melbourne, Vic 3010, Australia.
[4] Department of Economics, Southern Methodist University, Dallas, TX, 75275, USA.

1. Introduction.

In this paper we demonstrate methods for drawing inferences from estimated elasticities of demand. A significant literature in the estimation of demand relationships centers on the determination of elasticities. Such parameters of interest include: the Hicksian and Marshallian price elasticities of demand, the Allen and Morishima elasticities of substitution, income and expenditure elasticities defined for Engle curves, and long-run elasticities defined in dynamic models can be defined as nonlinear functions of the estimated parameters. In addition, although most demand specifications imply that these elasticities of interest vary by prices, income, or level of output, it is frequently the case that there is little attempt to draw inferences at more than a single point and for only one level of significance. In this paper we demonstrate how these bounds can be generalized to consider multiple values and how the implied cumulative density function of the estimated elasticity can be used to visualize the relationship between the level of significance and the inferences drawn.

In particular, this analysis focuses on the wide class of elasticities which are defined as ratios of estimated relationships. The principle method we use to construct these intervals is based on Fieller's method. The advantage of the Fieller method is that it generates a more general class of confidence intervals than can be obtained from the traditional (mean $\pm$ t $\times$ standard deviation) intervals or the standard resampling methods while still employing the usual asymptotic distributional assumptions. Although based on the assumption of asymptotic normality, the Fieller confidence intervals are constrained to be neither symmetric nor finite. We demonstrate how this method contrasts to the usual approximation techniques by constructing a cumulative distribution function of the relationship of interest so that one can observe how the confidence interval can be defined at various levels of significance.

The application considered in this paper is the estimation of Engel curves using a cross-section of household expenditures. Although a number of authors have proposed complex specifications for these relationships in most cases their main focus has been on the shape of the Engle curve. The inferences to be drawn from such features of the curve as whether the income elasticities are indicative of a change in the nature of the good from a normal to a luxury good based on the inflection of the Engle curve are typically not the objective. In the application presented here we use both parametric and semi-parametric Engle curves to illustrate our method for drawing inferences from the results of various methods for estimation.

Aside from the application of standard ordinary least squares regression we also employ two other parametric methods that are designed to account for the presence of a large proportion of zero demand values in one of the commodities under consideration. We also demonstrate the application of our method with the results from a semiparametric technique that allows for a nonparametric fit to the partial relationship between the shares of commodity expenditure to the total expenditure.

The paper proceeds as follows: In Section 2 we examine how the elasticity from a typical demand specification implies a ratio and some intuition into the nature of the Fieller method for the construction of confidence intervals and how it is related to the widely employed Delta approximation. In Section 3 we examine the particular case of the expenditure elasticities as defined from an Engel curve. In Section 4 we present the results of the estimation of the Engel curve using four methods. In Section 5 we conduct a comparison of the methods employed in Section 4 and our conclusions are presented in Section 6.

2. Elasticities and the Fieller Method.

In this section we first review how the usual point elasticity measure estimates require the construction of probability statements concerning the ratio of estimated parameters or functions of estimated parameters. We then review the Fieller method for the construction of confidence intervals and how the Fieller is related to the Delta method for the approximation of the standard error of a nonlinear function of estimated parameters. Finally we discuss how the Fieller interval can be interpreted as the solution to a constrained optimization which can be examined geometrically.

2.1 Elasticity Estimates and the Fieller Interval

The Fieller method for the estimation of confidence intervals for elasticities has been proposed by a number of authors. Fuller and Martin (1961) first propose the Fieller for the construction of intervals for the case of the dynamic elasticity and Fuller (1962) subsequently uses the Fieller to derive the confidence intervals of isoclines based on an estimated production function. Miller, Capps and Wells (1984) were the first to demonstrate how the Fieller could be widely employed for elasticities. This result was affirmed by Dorfman, Kling and Sexton (1990) with the addition of resampling methods in the comparison of techniques although the applications they considered resulted in less dramatic differences which may be due to some factors we discuss in Sections 2.3 and 2.4 below. Krinsky and Robb (1986, 1991) reject the application of the Delta approximation for elasticities however they do not consider the Fieller as a possible competitor to the bootstrap. Li and Maddala (1999) have less success with the Fieller for the computation of the long-run impact as measured in a dynamic model however, recently Bernard et al. (2007) have discovered that in the dynamic regression case the Fieller performs very well.

The primary case in which one may consider the elasticity as a ratio is the simple case of a demand specification of the form:

$$y_i = f(x_i) + \varepsilon_i \tag{1}$$

Where y is the quantity demanded and x is the variable of interest (i.e. income or price). Thus we use the definition of a point elasticity of y with respect to x evaluated at a particular value of $x = x_i$ to be

$$\eta_{y|x_i} = \left(\frac{\partial y(x_i)}{\partial x}\right)\frac{x_i}{y_i} \tag{2}$$

Or estimated as:

$$\begin{aligned}\hat{\eta}_{y|x_i} &= \left(\widehat{\frac{\partial y(x_i)}{\partial x}}\right)\frac{x_i}{\hat{y}_i}\\ &= \left(\widehat{\frac{\partial y(x_i)}{\partial x}}\right)\frac{x_i}{\hat{f}(x_i)}\end{aligned}$$

In a simple linear parametric case where $f(x) = \beta_0 + \beta_1 x$ the estimated partial derivative is $\left(\frac{\partial y(x_i)}{\partial x}\right) = \hat{\beta}_1$ and the estimated elasticity is defined as a ratio of linear combinations of the parameter estimates:

$$\hat{\eta}_{y|x_i} = \frac{\hat{\beta}_1 x_i}{\hat{\beta}_0 + \hat{\beta}_1 x_i} \tag{3}$$

A parallel literature in applied statistics that concerns a similar problem has appeared in the analysis of biological assay experiments. In the simplest version of this problem a logistic regression is fit to a series of observations in which differing levels of a substance (drug/poison) x is administered and determination of the result (curing/death) is recorded as the event with a binomial outcome. A linear equation for the log of the odds ratio can be specified as:

$$\ln\left(\frac{p(x_i)}{(1-p(x_i))}\right) = \beta_0 - \beta_1 x_i + \nu_i \tag{4}$$

where v_i is the error term and $p(x_i)$ is the probability of the event. From (4) we define the ratio of the intercept and the slope parameter $\left(\psi = \frac{\beta_0}{\beta_1}\right)$ as the 50% dose response level of x_i – where the probability of the event is just as likely to occur as not occur ($p(x_i) = .5$). In many applications the bounds on this critical value are of vital importance and Fieller (1944) provides a solution for the construction of the bounds on the estimate of $\hat{\psi} = \frac{\hat{\beta}_0}{\hat{\beta}_1}$ that has subsequently been widely used in this literature. In particular, Finney (1952, 1978) demonstrates with numerous examples the application of the Fieller method for this problem. More recent research into the properties of the application of the Fieller method has provided additional evidence of the practical advantages of the Fieller over alternative methods. Sitter and Wu (1993) conclude that the Fieller interval is generally superior to the Delta method.

2.2 The Fieller Interval

The application of the Fieller method for the construction of confidence intervals to ratios of the general case of linear combinations of regression parameters can be found in Zerbe (1978) and Rao (1973, page 241). Fieller's method in the general regression context is defined for the confidence interval for the ratio $\psi = \frac{\mathbf{K}'\boldsymbol{\beta}}{\mathbf{L}'\boldsymbol{\beta}}$ which is defined in terms of linear combinations of the regression parameters from the same regression $\mathbf{Y}_{T\times 1} = \mathbf{X}_{T\times k}\boldsymbol{\beta}_{k\times 1} + \boldsymbol{\varepsilon}_{T\times 1}$, $\boldsymbol{\varepsilon} \sim (\mathbf{0}_{T\times 1}, \boldsymbol{\Omega}_{T\times T})$. The FGLS estimators for the parameters are $\hat{\boldsymbol{\beta}} = (\mathbf{X}'\hat{\boldsymbol{\Omega}}^{-1}\mathbf{X})^{-1}\mathbf{X}'\hat{\boldsymbol{\Omega}}^{-1}\mathbf{Y}$, with a suitable estimate of $\boldsymbol{\Omega}$ and the vectors $\mathbf{K}_{k\times 1}$ and $\mathbf{L}_{k\times 1}$ are known constants. Under the usual assumptions we have that the parameter estimates are asymptotically normally distributed as $\hat{\boldsymbol{\beta}} \sim \mathbf{N}(\boldsymbol{\beta}, \boldsymbol{\Sigma})$ where $\hat{\boldsymbol{\Sigma}} = (\mathbf{X}'\hat{\boldsymbol{\Omega}}^{-1}\mathbf{X})^{-1}$. The 100(1-$\alpha$)% confidence interval for ψ is determined by solving the quadratic equation $a\psi^2 + b\psi + c = 0$, where t is the t-statistic for the $1-\alpha$ level of significance, $a = (\mathbf{L}'\hat{\beta})^2 - t^2\mathbf{L}'\hat{\boldsymbol{\Sigma}}\mathbf{L}$, $b = 2\left[t^2\mathbf{K}'\hat{\boldsymbol{\Sigma}}\mathbf{L} - (\mathbf{K}'\hat{\beta})(\mathbf{L}'\hat{\beta})\right]$, and $c = (\mathbf{K}'\hat{\beta})^2 - t^2\mathbf{K}'\hat{\boldsymbol{\Sigma}}\mathbf{K}$. The two roots of the quadratic equation $(\psi_1, \psi_2) = \frac{-b\pm\sqrt{b^2-4ac}}{2a}$, define the confidence bounds of the parameter value.

For example, in the case of the elasticity from the simple linear demand specification regression equation in (1) we define: $a = \left(\hat{\beta}_0 + \hat{\beta}_1 x_i\right)^2 - t^2\left(\hat{\sigma}_0^2 + 2x_i\hat{\sigma}_{01} + x_i^2\hat{\sigma}_1^2\right)$, $b = 2t^2\left(x_i\hat{\sigma}_{01} + x_i^2\hat{\sigma}_1^2\right) - \hat{\beta}_1 x_i\left(\hat{\beta}_0 + \hat{\beta}_1 x_i\right)$ and $c = \left(\hat{\beta}_1 x_i\right)^2 - t^2 x_i^2\hat{\sigma}_1^2$. Where $\hat{\boldsymbol{\Sigma}} = \begin{bmatrix} \hat{\sigma}_0^2 & \hat{\sigma}_{01} \\ \hat{\sigma}_{01} & \hat{\sigma}_1^2 \end{bmatrix}$. And the bounds given by the two roots:

$$(\psi_1, \psi_2) = \frac{\left(t^2\sigma_{01}x_i - \tfrac{1}{2}\beta_0\beta_1 x_i - \tfrac{1}{2}\beta_1^2 x_i^2 + t^2\sigma_1^2 x_i^2 \pm x_i\sqrt{\begin{matrix} t^4\sigma_{01}^2 - t^2\beta_0\beta_1\sigma_{01} - \tfrac{3}{2}\beta_0\beta_1^3 x_i - \tfrac{3}{4}\beta_0^2\beta_1^2 - \tfrac{3}{4}\beta_1^4 x_i^2 + t^2\beta_1^2\sigma_{01}x_i \\ + t^2\beta_0\sigma_1^2\beta_1 x_i + t^2\sigma_0^2\beta_1^2 + t^2\beta_0^2\sigma_1^2 - t^4\sigma_0^2\sigma_1^2 + t^2\sigma_1^2\beta_1^2 x_i^2 \end{matrix}}\right)}{2t^2\sigma_{01}x_i - \beta_0^2 - 2\beta_0\beta_1 x_i + t^2\sigma_0^2 - \beta_1^2 x_i^2 + t^2\sigma_1^2 x_i^2} \quad (5)$$

In order to have two real roots $a > 0$ (Buonaccori 1979). Besides the finite interval, the resulting confidence interval may be the complement of a finite interval ($b^2 - 4ac > 0$, $a < 0$) or the whole real line ($b^2 - 4ac < 0$, $a < 0$). These conditions are discussed in Scheffé (1970), Zerbe (1982) and Gleser and Hwang (1987). It can be shown that as long as the absolute value of the implied t-statistic for the denominator is greater in magnitude than the critical value for the confidence interval $\left|\frac{\mathbf{L}'\hat{\boldsymbol{\beta}}}{\sqrt{\mathbf{L}'\hat{\boldsymbol{\Sigma}}\mathbf{L}}}\right| > t$, both roots will be real. The intuition behind this result is as follows: for real roots we require that the denominator has sufficient probability mass away from zero. However, we also find that the implied t-statistic for the numerator also has a role to play in the formation of these intervals.

2.3 The Relationship between the Delta Interval and the Fieller Interval

The Delta method (see Rao 1973, Page 385) provides an approximate standard deviation for the ratio in this case we have

$$(\psi_1, \psi_2) = \hat{\psi} \pm t\left[\left(\frac{\mathbf{K}'\hat{\boldsymbol{\Sigma}}\mathbf{K}}{(\mathbf{L}'\hat{\boldsymbol{\beta}})^2}\right) + \hat{\psi}^2\left(\frac{\mathbf{L}'\hat{\boldsymbol{\Sigma}}\mathbf{L}}{(\mathbf{L}'\hat{\boldsymbol{\beta}})^2}\right) - 2\hat{\psi}\left(\frac{\mathbf{K}'\hat{\boldsymbol{\Sigma}}\mathbf{L}}{(\mathbf{L}'\hat{\boldsymbol{\beta}})^2}\right)\right]^{1/2} \qquad (6)$$

In the solution for the Fieller method CI we can rewrite the expression for a as $a = (\mathbf{L}'\hat{\beta})^2(1-g)$ where

$g = t^2\left(\frac{\mathbf{L}'\hat{\boldsymbol{\Sigma}}\mathbf{L}}{(\mathbf{L}'\hat{\boldsymbol{\beta}})^2}\right)$. It can be shown that the smaller the value for g the closer the Fieller and the Delta CIs become (see Finney (1952-page 63,1978- page 81), Cox (1990), Sitter and Wu (1993)). Note that this expression can be interpreted as the square of the critical value of the t-statistic for the 1-α confidence interval divided by the square of the implied t-statistic for the estimate of the denominator ($\mathbf{L}'\hat{\boldsymbol{\beta}}$). Thus the larger the t-statistic for the test of the hypothesis $H_0 : \theta = 0$ versus $H_1 : \theta \neq 0$, the smaller the value of g and the more similar the Delta and Fieller intervals become. Finney (1978 - page 82) suggests that a reasonable rule of thumb would be to use the Delta CI when the absolute value of the t-statistic for the denominator ($\hat{\theta}$) is 4 to 5 times greater than the t-statistic for the confidence bound ($g < .05$). Thus if we use $t = 2$ for $\alpha = .05$ the absolute value of the denominator t-statistic would be from 8 to 10. Sitter and Wu (1993) caution that such rules of thumb may overly simplify the choice of CI, specifically, when a researcher is interested in either only the upper or lower bound the Fieller may provide improved coverage over the Delta when g is less than .05. Furthermore, Herson (1975) demonstrates that when the covariance of the numerator to the denominator is positive (negative) the Delta and the Fieller intervals match more closely when the expected value of the ratio is positive (negative). These aspects become more obvious using the geometric representation given below.

2.4 A Geometric Representation of the Fieller Interval

Hirschberg and Lye (2007a) demonstrate that the Fieller interval is equivalent to the solution for ψ from the constrained optimization defined by the Lagrangian:

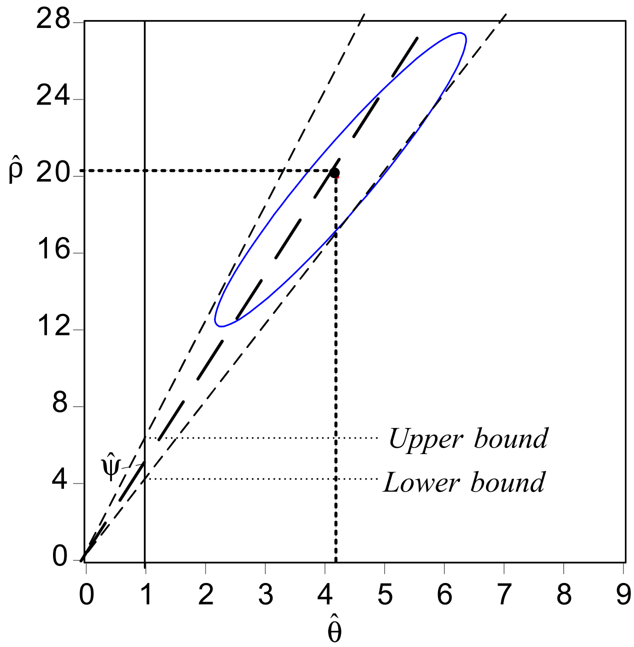
$$L(\psi,\lambda,\theta) = \psi - \lambda\left(\begin{bmatrix}(\hat{\rho}-\psi\theta) & (\hat{\theta}-\theta)\end{bmatrix}\begin{bmatrix}\hat{\omega}_{11} & \hat{\omega}_{12} \\ \hat{\omega}_{12} & \hat{\omega}_{22}\end{bmatrix}\begin{bmatrix}(\hat{\rho}-\psi\theta) \\ (\hat{\theta}-\theta)\end{bmatrix} - t^2\right) \qquad (7)$$

where we define the ratio as $\psi = \frac{\rho}{\theta}$, $\rho = \mathbf{K}'\boldsymbol{\beta}$, and $\theta = \mathbf{L}'\boldsymbol{\beta}$. With the quantities given from the estimation as $\hat{\rho} = \mathbf{K}'\hat{\boldsymbol{\beta}}$, $\hat{\theta} = \mathbf{L}'\hat{\boldsymbol{\beta}}$ and the estimated covariance of $\hat{\rho}$ and $\hat{\theta}$ is defined as:

$\begin{bmatrix} \hat{\omega}_{11} & \hat{\omega}_{12} \\ \hat{\omega}_{12} & \hat{\omega}_{22} \end{bmatrix} = \left[\begin{array}{c|c} \mathbf{K}'\hat{\Sigma}\mathbf{K} & \mathbf{K}'\hat{\Sigma}\mathbf{L} \\ \hline \mathbf{L}'\hat{\Sigma}\mathbf{K} & \mathbf{L}'\hat{\Sigma}\mathbf{L} \end{array}\right]$. Again t^2 is the square of the appropriate *t*-stat for the $1-\alpha$ level of significance (or the critical value of an *F*-distribution with 1 degree of freedom in the denominator).[5]

Note that the constraint in this case is the ellipse formed for those values that are consistent with a $1-\alpha$ probability for a linear combination of $\hat{\rho}$ and $\hat{\theta}$ to be significant.[6] This implies that we may show this optimization with a diagram of the quadratic constraint of a ray from the origin. In this way we can demonstrate the relationship between the bounds obtained from the Fieller and the nature of the joint distribution of $\hat{\rho}$ and $\hat{\theta}$.

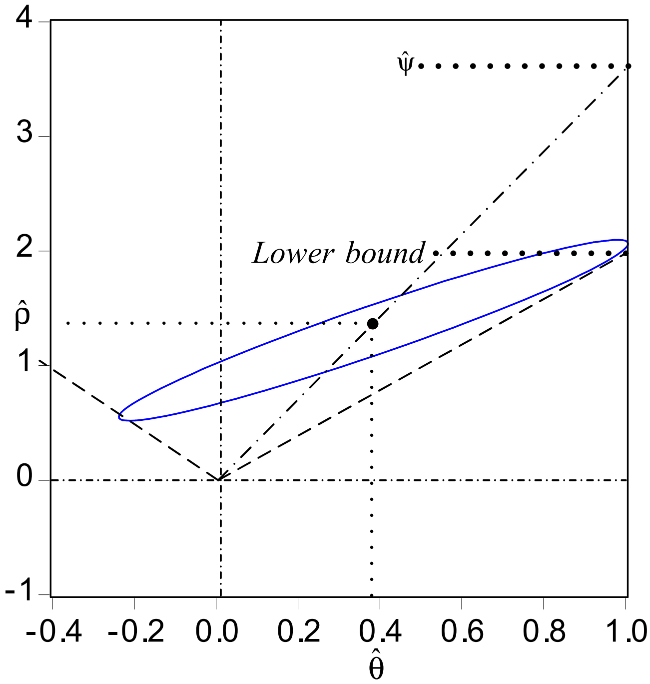
Figure 1: Finite Confidence Bounds from the Fieller.



5 See von Luxburg and Franz (2004) for an alternative optimization which has the same result.
6 Note that this is not the joint confidence bound for both random variables as often defined in textbooks which is similar in shape but larger in that it is typically limited by the critical value of an *F*-statistic with 2 degrees of freedom in the numerator or a Chi-square with 2 degrees of freedom.

Figure 1 demonstrates the case in which the Fieller method results in finite bounds.[7] The ratio $\hat{\psi} = \frac{\hat{\rho}}{\hat{\theta}}$ is the slope of the line through the points (0,0) and $\left(\hat{\rho}, \hat{\theta}\right)$. The two limiting rays from the origin define the $1-\alpha$ CI of ψ. We can read these bounds on the vertical axis at the point where these limiting rays intersect a line where the *x*-axis equals 1.

Figure 2: An Example where the lower bound is finite and the upper bound is infinite.



If the ellipse is located too close to the origin we will be unable to construct the two limiting rays from the origin to the edge of the ellipse. One possibility is shown in Figure 2. In this case $\hat{\theta}$, the denominator in the ratio, is not estimated with a high degree of precision. In fact the $1-\alpha$ confidence bound for the estimate includes zero. Note that the horizontal limits of the ellipse define the univariate $1-\alpha$ confidence bounds for θ which in this case results in a negative lower bound. This is the case when the *t*-statistic for the test of the hypothesis that $H_0 : \theta = 0$ is less than the critical value of the *t* used for the confidence interval of the ratio. The practical interpretation of this case is that the ratio has a finite lower bound but no upper bound.

In the case where the origin (0,0) is within the ellipse we are unable to construct tangents to the ellipse thus we have no real roots and our confidence interval encompasses the entire real line. This

[7] See Hirschberg and Lye (2007a) for details as to how standard econometric software can be used to generate these plots.

would be the case where the quadratic equation (such as (5)) to be solved for the Fieller interval does not possess real roots. Note that by changing the critical value we change the area of the ellipse. Thus it may be the case that although neither bound is finite when $\alpha = .05$ we may find either one or both are finite for $\alpha = .10$. Conversely, if we can define a finite interval for $\alpha = .05$ we may not for $\alpha = .01$. And for any case the Fieller interval becomes infinite before $\alpha = 0$. Also note that as the ellipse moves further away from the origin the value of *g* defined above becomes smaller and the Delta and Fieller CIs become equivalent. And that the correlation between $\hat{\rho}$ and $\hat{\theta}$ will influence the angle of the major axis of the ellipse which will influence the symmetry of the bounds.

2.5 Comparing Alternative Methods.

A number of studies have compared the Fieller method confidence intervals with the alternative methods for the construction of intervals for the ratio of means. Monte Carlo experiments to assess the performance of a number of different methods have been performed by Jones et al. (1996) for statistical calibration, Williams (1986) and Sitter and Wu (1993) in bioassay, Polsky et al. (1997), Briggs et al. (1999) for cost-effectiveness ratios, Freedman (2001) for intermediate or surrogate endpoints and Hirschberg and Lye (2005) for the extremum of a quadratic regression.

Generally, the results from these Monte Carlo simulations indicate that the Fieller-based methods work reasonably well under a range of assumptions including departures from normality. The Delta-based method is a consistent poor performer and often underestimates the upper limit of the intervals. They have concluded that the Fieller method is superior to the traditional Delta method based on a first order approximation of the variance of the ratio and the traditional symmetric confidence bounds. Alternative methods based on resampling methods such as the bootstrap, Bayesian methods and the inverse of the likelihood ratio test have all been compared in different simulations.[8] In general, it has been found that the Fieller method is as efficient to compute and results in comparable coverage to all these other methods. The analysis shown in Sections 2.3 and 2.4 can be used to demonstrate that simulations in which the joint distribution of the numerator and denominator are located far from the

[8] The inverse of the likelihood ratio test is asymptotically equivalent to the Fieller however is based on Chi-square distribution as opposed to the *t*-distribution, thus in small samples differences may be more pronounced.

origin will result in finding little difference between the Fieller and alternative confidence intervals. However, in simulations where the joint distribution approaches the origin it is found that the Fieller often dominates or is a close equal to most other more difficult methods to use.

3. Engel Curve Estimation Using Household Expenditure Data

In order to demonstrate how the Fieller method can be used for the construction of confidence intervals for elasticities we present an application in which we apply a series of different estimation methods and demonstrate how the results of these elasticity estimates can be portrayed. The Engel curve is the relationship between the amount of a good purchased and income. The specification of the Engel curves that we estimate is typical of the methods employed when using household expenditure survey data. These data record the level of expenditures by item and service for a household along with a series of demographic characteristics. Thus the specification of the Engel curve is based on levels of expenditures and not on the quantity since the assumption of a unit price can not be made for most commodities in the survey. Also due to the various difficulties in defining income for a household total expenditure by the household is frequently employed as the proxy for household income.

The specification is defined by the share as a function of the log of the total expenditure:

$$y_i = g(\ln(c_i), x_i) + \varepsilon_i \qquad (8)$$

where y_i is the expenditure share on the commodity or service by household *i*, c_i is the total expenditure by household *i*, and x_i are the household characteristics of household *i*. The elasticity for a particular household type[9] *j* is defined as:

$$\eta_{y|c_j} = \frac{\left(g(\ln(c_j), x_j) + \left(\frac{\partial g(\ln(c_j), x_j)}{\partial \ln(c)}\right)\right)}{g(\ln(c_j), x_j)} \qquad (9)$$

Once an Engel curve relationship has been estimated the elasticity is estimated by:

[9] We define household types as all having the average demographic characteristics and different levels of total expenditure.

$$\hat{\eta}_{y|c_j} = \frac{\hat{\rho}_j}{\hat{\theta}_j} \tag{10}$$

where $\hat{\rho}_j = \hat{g}(\ln(c_j), x_j) + \left(\frac{\partial \hat{g}(\ln(c_j), x_j)}{\partial \ln(c)}\right)$ and $\hat{\theta}_j = \hat{g}(\ln(c_j), x_j)$. Thus the elasticity estimates are formed by a ratio of the predicted share plus the first derivative of the share with respect to the log of expenditure divided by the predicted share.

The specification of $g(\ln(c_j), x_j)$ is either a parametric or a semi-parametric form with an error defined as either non-bounded or censored. These models have been fit using traditional regression, Tobit or censored regression – to account for zero-valued expenditures for some items, robust regression - to account for the presence of outliers in household data and zero valued dependent values and by the use of semi-parametric models.

3. 1 Parametric Specifications

The parametric specification used in the applications presented here is a general form to allow for flexibility that embeds the traditional quadratic as well as allowing for more flexibility by the use of a 2nd order Laurent expansion as proposed by Barnett(1983). We define the specification as:

$$y_i = \alpha_0 + \sum_{k=1}^{K} \alpha_k x_{ik} + \left(\gamma_1 \ln(c_i) + \gamma_2 \ln(c_i)^2 + \gamma_3 \ln(c_i)^{-1} + \gamma_4 \ln(c_i)^{-2}\right) + \varepsilon_i \tag{11}$$

where the x_k are K demographic characteristics of the household which we would like to control for. The estimation of this function can be estimated as a linear equation.

In this case the estimate of the partial derivative of the expenditure with respect to the log of total expenditure is given by a linear combination of the estimated parameters:

$$\frac{\partial \hat{g}(\ln(c_j), x_j)}{\partial \ln(c)} = \hat{\gamma}_1 + \hat{\gamma}_2\left[2\ln(c_j)\right] + \hat{\gamma}_3\left[-\ln(c_j)^{-2}\right] + \hat{\gamma}_4\left[-2\ln(c_j)^{-3}\right] \tag{12}$$

The predicted value $\hat{g}(\ln(c_j), x_j)$ is another linear combination of the parameter estimates, thus $\hat{\eta}_{y|c_j} = \frac{\hat{\rho}_j}{\hat{\theta}_j}$ and in this case:

$$\hat{\rho}_j = \hat{\alpha}_0 + \sum_{k=1}^{K} \hat{\alpha}_k x_{jk} + \begin{pmatrix} \hat{\gamma}_1\left[\ln(c_j) + 1\right] + \hat{\gamma}_2\left[\ln(c_j)^2 + 2\ln(c_j)\right] \\ +\hat{\gamma}_3\left[\ln(c_j)^{-1} - \ln(c_j)^{-2}\right] + \hat{\gamma}_4\left[\ln(c_j)^{-2} - 2\ln(c_j)^{-3}\right] \end{pmatrix}$$

and:

$$\hat{\theta}_j = \hat{\alpha}_0 + \sum_{k=1}^{K} \hat{\alpha}_k x_{jk} + \left(\hat{\gamma}_1 \ln(c_j) + \hat{\gamma}_2 \ln(c_j)^2 + \hat{\gamma}_3 \ln(c_j)^{-1} + \hat{\gamma}_4 \ln(c_j)^{-2} \right)$$

Once the estimated covariance matrix of the parameters has been estimated then we can determine the Fieller interval for the elasticity evaluated for any household defined by the levels of the regressors.

One exception to this is when the estimates are generated via a Tobit or other truncated regression technique. In this case the predicted value and the marginal impact of expenditure are not simple linear functions of the parameters but must also account for the probability model assumed. In these applications we will use the Tobit or Normit model which assumes that the data are normally distributed with a truncation point at zero.[10] Using the results from McDonald and Moffitt (1980) we define the unconditional estimated expected value of the share for household type j as:

$$\hat{E}[y_j] = \Phi\left(\hat{y}_j / \hat{\sigma}\right)\hat{y}_j + \hat{\sigma}\phi\left(\hat{y}_j / \hat{\sigma}\right) \tag{13}$$

where $\hat{y}_j = \hat{g}(\ln(c_j), x_j)$. In this case we assume the error in the regression equation specified in (8) is defined as $\boldsymbol{\varepsilon} \sim \mathrm{N}(\mathbf{0}, \sigma^2\mathbf{I})$, $\Phi(z)$ and $\phi(z)$ are the cumulative normal function and the normal density function evaluated at z. The derivative of the unconditional expected value of y with respect to ln(x) is estimated by:

$$\frac{\partial \hat{E}\left[y_j\right]}{\partial \ln(c)} = \frac{\partial \hat{g}(\ln(c_j), x_j)}{\partial \ln(c)} \Phi\left(\hat{y}_j / \hat{\sigma}\right) \tag{14}$$

Thus the estimated unconditional elasticity would be defined as :

$$\hat{\eta}_{y|c_j} = \frac{\hat{\rho}_j}{\hat{\theta}_j} \tag{15}$$

where $\hat{\theta}_j = \Phi\left(\hat{y}_j / \hat{\sigma}\right)\hat{y}_j + \hat{\sigma}\phi\left(\hat{y}_j / \hat{\sigma}\right)$ and $\hat{\rho}_j = \hat{\theta}_j + \left(\frac{\partial \hat{g}(\ln(c_j), x_j)}{\partial \ln(c)}\right)\Phi\left(\hat{y}_j / \hat{\sigma}\right)$. When the model is a parametric model as defined as in (11) $\hat{\rho}_j = \hat{\theta}_j + \left\{\hat{\gamma}_1 + \hat{\gamma}_2\left[2\ln(c_j)\right] + \hat{\gamma}_3\left[-\ln(c_j)^{-2}\right] + \hat{\gamma}_4\left[-2\ln(c_j)^{-3}\right]\right\}\Phi\left(\hat{y}_j / \hat{\sigma}\right)$.

Alternatively, the conditional estimated expected value of the share for the case $y > 0$ is:

[10] In application used here we ignore the possible upper censoring of the shares at one because we do not have all the shares for the households in our sample.

$$\hat{E}[y_j \mid y > 0] = \hat{y}_j + \hat{\sigma}\lambda\left(\hat{y}_j / \hat{\sigma}\right) \tag{16}$$

where $\lambda(z) = \left(\frac{\phi(z)}{\Phi(z)}\right)$ is the inverse Mills ratio. The derivative of the conditional expected value of y with respect to $\ln(x)$ is given by:

$$\frac{\partial \hat{E}\left[y_j|y>0\right]}{\partial \ln(c)} = \frac{\partial \hat{g}(\ln(c_j), x_j)}{\partial \ln(c)}\left[1 - \left(\hat{y}_j / \hat{\sigma}\right)\lambda\left(\hat{y}_j / \hat{\sigma}\right) - \lambda\left(\hat{y}_j / \hat{\sigma}\right)^2\right] \tag{17}$$

Here we will refer to the conditional estimate of the elasticity when $y > 0$ as $\tilde{\eta}_{y|c_j}$ and it is defined as:

$$\tilde{\eta}_{y|c_j} = \frac{\tilde{\beta}_j}{\tilde{\theta}_j} \tag{18}$$

where $\tilde{\theta}_j = \hat{y}_j + \hat{\sigma}\lambda\left(\hat{y}_j / \hat{\sigma}\right)$ and $\tilde{\beta}_j = \tilde{\theta}_j + \frac{\partial \hat{E}\left[y_j|y>0\right]}{\partial \ln(c)}$. When the model is specified as the parametric form defined in (11) we would use:

$$\tilde{\beta}_j = \tilde{\theta}_j + \left\{\hat{\gamma}_1 + \hat{\gamma}_2\left[2\ln(c_j)\right] + \hat{\gamma}_3\left[-\ln(c_j)^{-2}\right] + \hat{\gamma}_4\left[-2\ln(c_j)^{-3}\right]\right\}\left[1 - \left(\hat{y}_j / \hat{\sigma}\right)\lambda\left(\hat{y}_j / \hat{\sigma}\right) - \lambda\left(\hat{y}_j / \hat{\sigma}\right)^2\right].$$

4. Estimation methods for the Engel curve

The estimation of Engel Curves has been proposed using a number of techniques. Historically the primary method has been the application of a parametric model similar to the one specified in (11). Alternatively, semi-parametric models that allow for any functional form relationship between the budget shares and total expenditures, but assume that the demographic variables enter the model in a linear way, have been used (see eg. Blundell, 1998; Bhalotta and Attfield, 1998; Alan et al., 2002; Gong et al., 2005). An alternative approach employs quantile regression (see eg. Deaton, 1997; Koenker and Hallock, 2001), where the 50th quantile is the least absolute deviations estimator. It has been suggested that quantiles are resistant statistics (Davison 2003), that is, they are robust to outliers and contamination.

A typical characteristic for some commodities, such as education expenditure, is that a significant proportion of household observations are reported with zero expenditure. A number of explanations have been proposed for observed zero expenditure in the data. These include false reporting, infrequent purchases or non purchases. One approach has been to estimate using the entire sample irrespective of whether households had zero or positive expenditure on a particular commodity. In other studies, semi-parametric regression has been used (see eg. Bhalotta and Attfield, 1998; Gong et al., 2005). Alternatively, limited dependent variable models have been proposed. One approach is to use the Tobit model (see eg Tansel and Bircan 2006) which assumes that the same set of variables determine both the probability of a non-zero consumption and the level of expenditure. A modification of the Tobit model is the double-hurdle model which is a two equation model with a binary choice part explaining the participation decision and a conditional regression equation explaining positive expenditure levels (see eg Cragg 1971; Melenberg and Van Soest 1996). Deaton and Irish (1984) also propose a number of models to account for misreporting of households. An alternative approach suggested by You (2003) is to assume that the exact source of zero expenditures is unobservable and include all the observations in the sample and to use robust estimation to deal with the problem of potential outliers and zero expenditures. Beatty (2007), on the other hand, has suggested that quantile methods may be a useful tool in dealing with zero expenditure.

In this paper we will demonstrate the estimation of Fieller intervals for the elasticity of expenditure share with respect to total expenditure based on the results of 4 different types of estimation procedures. The data used in this paper comes from the application by Gong, Van Soest, and Xhang (2005, henceforth GVS) which consists of expenditure data collected from a survey of rural Chinese communities entitled "Rural Household Income and Expenditure Survey' conducted by the State Statistics Bureau of China and the Chinese Academy of Social Science. The data was collected in 1995 and provides information on households from 19 Chinese provinces. Only using data from households with 2 parents and 1 or more children and excluding observations with missing or implausible values gives a sample of 5394 households for estimation. Table A.1 in the Appendix displays the sample statistics for these data. Also in the Appendix is Figure A.1 which provides a kernel density plot of log total expenditure.

The applications in GVS employ both a linear specification with a quadratic term for the influence of total expenditure as well as a semiparametric application. In following their analysis we will use their specification with a slight modification to allow for the linear and squared inverse terms. In addition, in their estimation GVS allow for the endogeneity of total expenditure. However, tests indicate that exogeneity of total expenditure is only rejected for alcohol and tobacco, and even in this case their estimates obtained are similar to those without correction for endogeneity. Thus, in our applications that follow we will not consider the possibility that total expenditure is endogenous, although the applications can all be readily extended to this case.

4.1 OLS Estimation

The first model we fit is the linear regression model as specified in (11) using the same demographic variables used by GVS using the ordinary least squares approach where the errors are assumed to be $\boldsymbol{\varepsilon} \sim (\mathbf{0}_{T\times 1}, \sigma^2 \mathbf{I}_T)$. From Table 1 it can be seen that the results are similar to those reported by GVS in their table III (page 519).

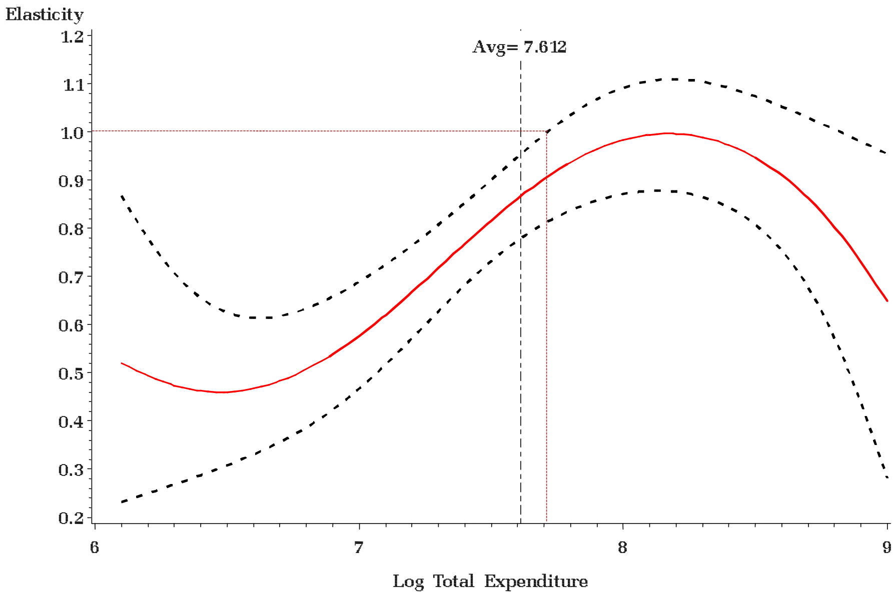
Table 1. The results of the OLS estimation as applied to the GVS data.

	Food		*Education*		*Alc & Tob*	
	Coefficient	*se*	*Coefficient*	*se*	*Coefficient*	*se*
Intercept	14246.02	5251.37	-1169.66	773.14	-1619.52	720.03
AG	-58.04	22.57	2.33	2.85	1.73	3.09
AG2/100	66.43	26.10	-3.07	3.05	-4.99	3.58
DCOAST	-1.22	0.59	0.14	0.09	0.56	0.08
DMIDDLE	-6.60	0.52	0.03	0.08	0.23	0.07
CHILD6	-6.66	4.28	0.26	0.48	-1.24	0.59
GIRL6	0.21	3.93	-0.49	0.58	-0.02	0.54
*CHILD12/PUP12**	-5.80	3.27	0.66	0.28	-0.99	0.45
*GIRL12/PUG12**	-0.45	2.20	-0.12	0.35	-0.17	0.30
*CHILD15/PUP15**	-4.92	3.41	1.42	0.36	-0.60	0.47
*GIRL15/PUG15**	-8.75	3.21	-0.08	0.50	-1.02	0.44
*CHILD18/PUP18**	-8.35	3.36	2.42	0.48	-1.02	0.46
*GIRL18/PUG18**	0.56	3.23	-1.76	0.70	-0.23	0.44
PADU	-2.09	2.22			0.41	0.30
PFADU	-1.61	2.30			-0.69	0.32
NUM	-1.82	0.29	0.03	0.04	-0.13	0.04
PUP19			4.80	0.80		
PUG19			-3.89	1.22		
LNCPER	-1310.38	466.68	107.88	68.71	149.42	63.99
LNCPER2	43.57	15.47	-3.68	2.28	-5.09	2.12
ILNCPER	-65878.99	26130.62	5551.07	3847.05	7695.11	3582.84
ILNCPER2	112524.05	48521.47	-9704.71	7143.39	-13388.85	6652.91
$\bar{R}^2$	.3649		.0210		.0314	

* In following GVS for the case of Education these variables are defined as the PUP version – the proportion of Children in school of this gender and age.

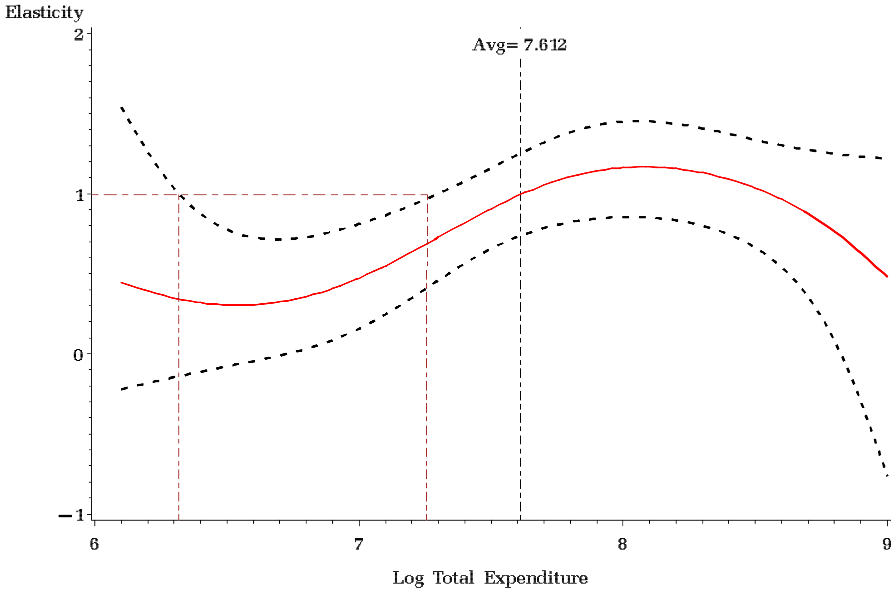
Using these estimates we determined the elasticity for the average household across the range of log total expenditure from 6 to 9 (note the mean is 7.6). Thus we can compute the elasticity for each set of values as well as the distribution of the estimate based on Fieller's method. In Figures 3, 4 and 5 we have the plots of the elasticity on the vertical axis, the log of total expenditure (LNCPER) on the horizontal axis, and a dashed line for the average log of total expenditure.

Figure 3 The elasticity of the Expenditure share for alcohol and tobacco with respect to the log of total expenditure with 95% Fieller confidence intervals based on OLS results.



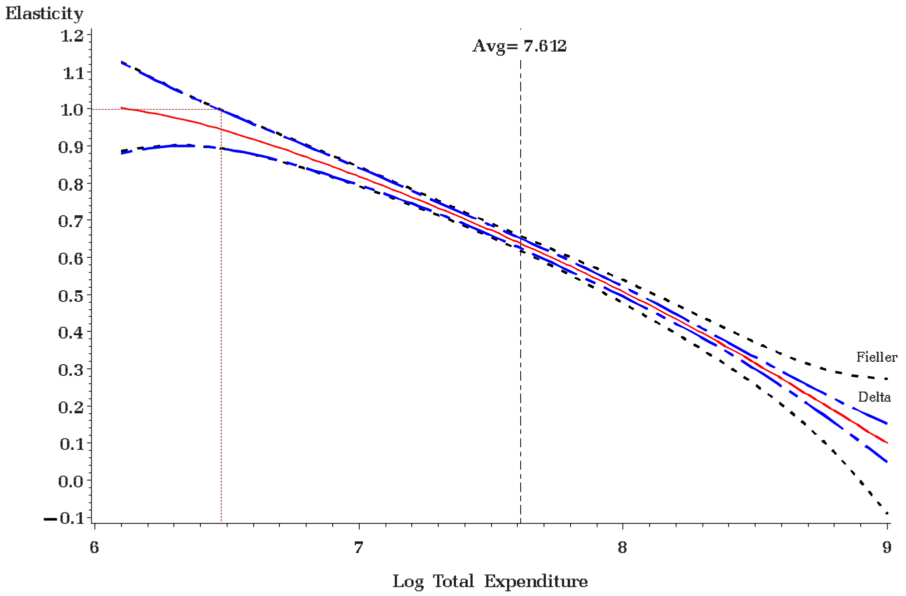
From Figure 3 we note that that the expenditure share on Alcohol and Tobacco is inelastic for average households with less than 7.7 log total expenditure and not different from 1 at expenditures above.

Figure 4 The elasticity of the expenditure share for education with respect to the log of total expenditure with 95% Fieller confidence intervals.



From Figure 4 we find that we are unable to reject the null hypothesis $H_0 : \eta_{y|c_j} = 1$ when $\alpha = .05$ except for a range of log expenditures between 6.3 and 7.3.

Figure 5 The elasticity of the expenditure share for food with respect to the log of total expenditure with 95% Fieller confidence intervals (small dash) as well as the Delta method (long dash) 95% confidence interval .



In Figure 5 it can be noted that for most of the range of values of the log of total expenditure the demand for food is income inelastic. In Figure 5 we have added the estimated 95% confidence interval estimated using the Delta as the Fieller method interval. Note that for the lower values of total expenditure the two intervals coincide quite closely and it is only near the top values the total expenditure data where the Delta interval is much smaller than the Fieller. However, if we compute the elasticity for education expenditures for low levels of total expenditure at which almost none of the sample buys education services we get a very different relationship between the two methods as is seen in Figure 6.

Figure 6 The lower levels of the elasticities for education expenses with the Delta (long dash) as well as Fieller (short dash) 95% CIs.

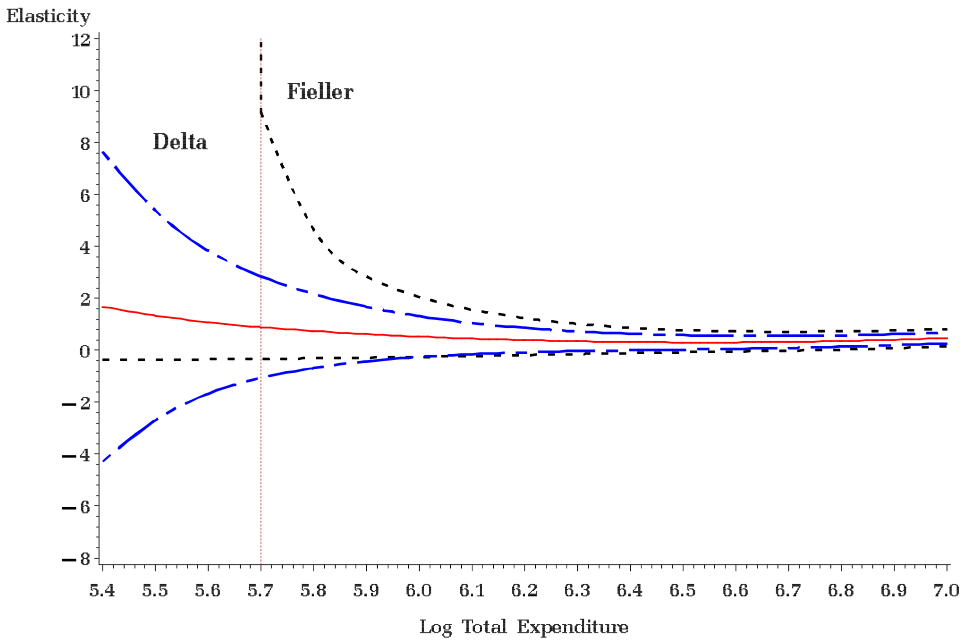
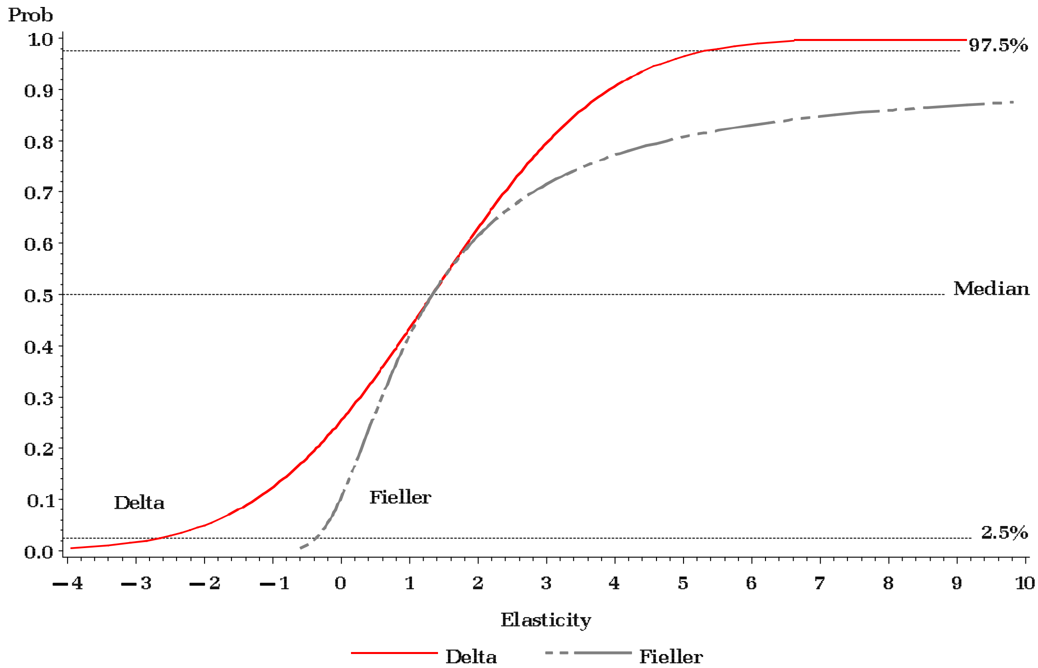


Figure 6 provides the same comparison of confidence intervals as in Figure 5 for education and where the log total expenditure ranges from 5.4 to 7. At log total expenditure values below 5.7 the upper Fieller bound is infinite while the lower bound is much smaller in magnitude than the corresponding Delta method interval which maintains the symmetry. This example demonstrates quite clearly why in a number of Monte Carlo studies which compare the Delta and the Fieller methods, the Delta method has been found to have comparable coverage to the Fieller in some cases and not others. In this example we note that although the upper bound has become infinite for the Fieller the lower bound is much higher than the bound estimated by the Delta method.

An alternative method for comparing these intervals is to define the implied cumulative density function (CDF) for the two methods. This is done by setting the value of $\alpha = .5$ then finding the implied upper and lower bounds for progressively smaller and smaller values of α. For the Fieller interval we eventually find a value of α where the bounds become infinite before $\alpha \to 0$. Figure 7 is a comparison of the CDFs computed for the elasticity for education for a log total expenditure level = 5.5 using the Delta approximation and the Fieller method. The CDF for the Delta demonstrates the typical shape of a CDF for a symmetrical distribution. The 95% confidence interval can be read from this graph by the values of the elasticity on the horizontal axis where the 2.5% and 97.5% lines cut the Delta CDF. In this case the interval is approximately -2.6 to 5.4. This interval could also be found from Figure 6 by drawing a vertical line from 5.5 which would cut the Delta 95% CI at the same values.

Figure 7 also shows the CDF for the Fieller interval, note that the two CDFs only coincide when the elasticity is equal to the ratio of the expected values at approximately 1.2 which is the 50th percentile or the median of the ratio. The CDF for the Fieller interval exhibits a significant degree of asymmetry and the significance level of the confidence bound at which the upper bound approaches infinity can be seen to be a bit less than where $\alpha = .2$ for the two sided test and .1 for the one sided test for the upper limit. Also note that if we are interested in only the lower 2.5% bound of the elasticity the Fieller interval implies a lower bound of approximately -.3 versus the Delta lower bound of approximately -2.6. In this case the Fieller has a much tighter lower bound than upper bound.

Figure 7 The CDF implied by the Fieller (dashed line) and Delta (solid line) confidence intervals for the elasticity of the expenditure on education when the log of total expenditure is 5.5



4.2 Robust Regression Estimation

An alternative estimation method to the usual OLS method for Engel curves is the use of a robust regression method. Typically in using expenditure survey data we find that there are a large number of outlier values both in the share of expenditure on particular commodities (such as alcohol and tobacco) and for some commodities (such as education) there may be a large number of observations where the dependent variable is zero. Because both these anomalies are present in the data used here (see GVS for more detail) we consider the application of a robust estimator. In addition, the application of a robust estimation procedure has the advantage of providing estimates and asymptotic standard errors of the parameters that we can use in a similar fashion to the standard regression results. Thus to compute the confidence intervals and the elasticities we can use the same techniques as we used in the case of the regression. In this example we follow You (2003) who found that in an analysis of Canadian expenditure data the MM estimator introduced by Yohai(1987) performed consistently better than competing robust regression methods.

The least trimmed estimate proposed by Rousseeuw (1984) is used for the initial estimates of the parameter vector prior to the application of the MM estimation procedure. The estimates of the covariance matrix are based on the reweighed $(\mathbf{X}'\mathbf{X})^{-1}$ matrix as defined by Huber (1981, page 173). The specification of the model is the same as used in the regression case. The results of the estimation are given below:

Table 2. The results of the robust estimation as applied to the GVS data.

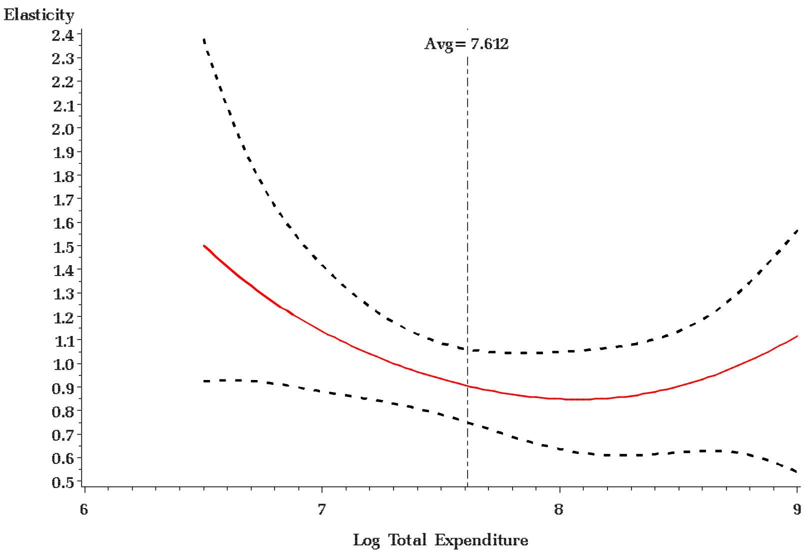
	Food		***Education***		***Alc & Tob***	
	Coefficient	***se***	***Coefficient***	***se***	***Coefficient***	***se***
Intercept	20909.05	5373.29	45.75	75.69	-236.64	435.47
AG	-60.53	24.12	-0.080	0.292	1.91	1.89
AG2/100	67.76	27.75	-0.071	0.312	-4.26	2.18
DCOAST	-0.69	0.63	-0.003	0.009	-0.02	0.05
DMIDDLE	-6.56	0.57	0.003	0.008	0.07	0.04
CHILD6	-6.83	4.62	-0.030	0.049	-0.83	0.36
GIRL6	-1.40	4.23	-0.058	0.059	0.11	0.33
*CHILD12/PUP12**	-6.43	3.50	0.079	0.030	-0.62	0.28
*GIRL12/PUG12**	-0.75	2.36	0.054	0.038	-0.10	0.18
*CHILD15/PUP15**	-5.62	3.63	0.070	0.039	-0.72	0.29
*GIRL15/PUG15**	-9.00	3.43	0.096	0.056	-0.03	0.27
*CHILD18/PUP18**	-7.91	3.60	0.037	0.053	-0.71	0.28
*GIRL18/PUG18**	-2.18	3.46	0.011	0.077	-0.21	0.27

	Food		Education		Alc & Tob	
	Coefficient	*se*	*Coefficient*	*se*	*Coefficient*	*se*
PADU	-2.75	2.38			0.22	0.19
PFADU	-1.38	2.47			-0.49	0.19
NUM	-1.83	0.31	0.001	0.004	-0.06	0.02
PUP19			0.207	0.086		
PUG19			-0.099	0.131		
LNCPER	-1927.92	478.00	-4.42	6.73	20.99	38.82
LNCPER2	64.70	15.86	0.16	0.22	-0.69	1.29
ILNCPER	-97386.28	26705.58	-202.98	376.08	1182.70	2160.32
ILNCPER2	167775.62	49522.83	325.81	697.07	-2124.21	3999.21
$\bar{R}^2$	.3305		.0229		.0048	

* In following GVS for the case of Education these variables are defined as the PUP version – the proportion of children in school of this gender and age instead of the proportion of all children in this age and gender group.

From Table 2 we note that the log total expenditure terms are estimated for alcohol & tobacco and for education with much less accuracy than was the case with the OLS result. However a test based on the differences in the sums of the squared error of the restricted versus the unrestricted model rejects the hypothesis that the coefficients estimated for the log of total expenditure terms are all equal to zero at the .01 level. Figure 8 shows how the elasticity for education for the mean household characteristics, varies by the log of total expenditure. Note that based on the robust estimation there is no level of total expenditure for which we can reject the hypothesis that the elasticity is not equal to one and there is also no level at which the elasticity is not significantly greater than zero. The plots for the other commodities based on this model are shown in Section 5 below.

Figure 8 The estimated elasticity using robust regression of the expenditure share for education with respect to the log of total expenditure with 95% Fieller confidence intervals.



4.3 The Tobit model for estimation

Due to the level of detail of an expenditure survey many households record zero for the consumption of a particular commodity. In the present sample 3031 of 5394 households reported expenditure levels for education as zero. Tobin's (1958) original application of the subsequently named Tobit model was in a demand context that related to household level data used for the estimation of Engel curves. A search of recent literature finds well over 100 papers that use a Tobit type model in the estimation of Engel curves. The estimation of regressions using censored data can be formulated using a number of different distributional assumptions. However the most common applications are based on the Normal distribution.

As we note above the estimated elasticity in the case of the Tobit model is defined as either the unconditional case $\hat{\eta}_{y|c_j} = \frac{\hat{\rho}_j}{\hat{\theta}_j}$ or the conditional case $\tilde{\eta}_{y|c_j} = \frac{\tilde{\rho}_j}{\tilde{\theta}_j}$. Once we have estimated the Tobit model using a standard maximum likelihood routine we also obtain an estimate for the asymptotic covariance matrix. Thus we might proceed to estimate the confidence bounds for the estimated elasticity in the same manner as in the case of the regression results. However, because $\hat{\rho}_j$, $\tilde{\rho}_j$, $\hat{\theta}_j$, and $\tilde{\theta}_j$ are defined as functions of the cumulative normal density function $\Phi(\hat{y}/\hat{\sigma})$, the normal density function $\phi(\hat{y}/\hat{\sigma})$ as well as of the parameter estimates we incur more complication to the estimation. In order to generate a Fieller interval we will use a bootstrap to estimate the variance covariance matrix of $\hat{\rho}_j$, $\tilde{\rho}_j$, $\hat{\theta}_j$, and $\tilde{\theta}_j$. Efron (1982 ch 5) and Efron and Tibshirani (1993 ch 6) discuss the use of the bootstrap to estimate the standard error and the covariance for statistics. Here we apply what is sometimes referred to as an unconditional bootstrap to the household data and re-run the regressions multiple times. In the unconditional bootstrap we resample the rows of the entire data set to create a series of pseudo-samples of the same size which can then be used to reestimate the regression relationship multiple times.

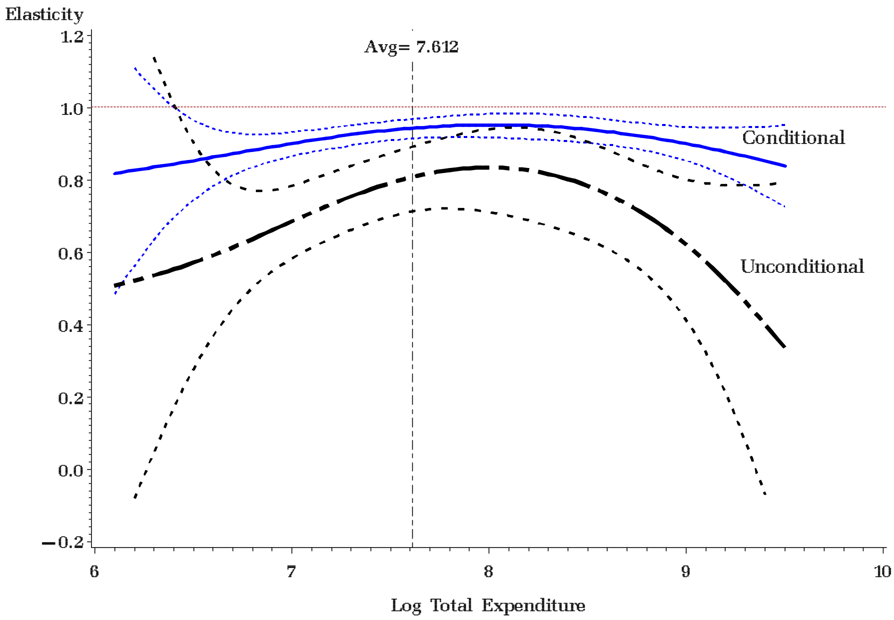
Table 3 lists the parameter estimates based on the application of the Tobit model to the GVS data for education expenditures the only expenditure item in this data for which more than half the dependent value is given as zero. Note that from these results all the log total expenditure variables are significant unlike the parameters estimated by the robust and OLS estimates.

Table 3. The results of the Tobit estimation as applied to the GVS data.

	Coefficient	***se***
Intercept	-1120.90	128.34
AG	5.13	5.94
AG2/100	-9.17	6.45
DCOAST	0.07	0.17
DMIDDLE	0.13	0.15
CHILD6	-0.14	0.93
GIRL6	-0.43	1.15
PUP12	1.75	0.53
PUG12	0.38	0.66
PUP15	3.27	0.68
PUG15	0.61	0.94
PUP18	4.63	0.90
PUG18	-2.46	1.32
NUM	0.12	0.07
PUP19	8.55	1.48
PUG19	-5.45	2.30
LNCPER	100.49	17.53
LNCPER2	-3.34	0.78
ILNCPER	5444.93	282.38
ILNCPER2	-9769.97	165.53
$\hat{\sigma}$	3.71	0.06

In order to determine the variance and covariance of $\hat{\rho}_j$, $\tilde{\rho}_j$, $\hat{\theta}_j$, and $\tilde{\theta}_j$ we computed their values using a first-order balanced bootstrap (see Davison and Hinkley (1997 page 439)) which insures that each observation is selected exactly *B* times, where *B* is the number of bootstrap replications was set to 1000. We then reestimated the Tobit model for the education data using a maximum likelihood estimation routine where the number of iterations was constrained to be 10 or less (the estimation of the complete sample in this case required more than 180 iterations). In this case we follow the recommendation of Davison and MacKinnon (1999) who propose that when bootstrapping the results of a MLE such as the Tobit, it is unnecessary to allow the process to converge completely for each bootstrap replication.

Figure 9 The estimated unconditional (broken line) and conditional (solid line) elasticities of the expenditure share for education with respect to the log of total expenditure with 95% Fieller confidence intervals.



The conditional and unconditional elasticities and the 95% Fieller bounds for education expenses are shown in Figure 9. From this figure it is readily apparent that the conditional elasticities lie above the unconditional elasticities at all observed levels of total expenditure and that the estimated precision of the conditional estimate is much greater than the corresponding unconditional estimates. In addition, by comparing Figure 9 with Figures 4 and 8 we can conclude that the unconditional and conditional elasticity estimates for education are markedly more precise than our findings from the OLS and robust regression. From Figure 9 we can conclude that for most levels of total expenditure both elasticities are significantly greater than zero and less than one.

4.4 The Semiparametric Regression Model.

GVS propose the use of a semiparametric Engel curve model that does not rely on the assumption of a parametric functional form such as (11). An alternative to the use of a parametric function is the use of a model that allows for the specification of a general function for the relationship between the expenditure share and the total expenditure level. This model would be specified as:

$$\begin{aligned} y_i &= g(\ln(c_i), x_i) + \varepsilon_i \\ &= \alpha_0 + \sum_{k=1}^{K} \alpha_k x_{ik} + h(\ln(c_i)) + \varepsilon_i \end{aligned}$$

where $h(\ln(c_i))$ is specified as a general function with a shape that is determined by the data and the x's are the demographic variables that determine the location by a linear model. In this example we will employ a penalized least squares method where a thin-plate quadratic smoothing spline is used to approximate the function $h(\ln(c_i))$. The estimation of such penalized splines via mixed or error component regression methods has been shown to be a fairly simple extension of the estimation of linear mixed regression model estimation by Ruppert, Wand and Carroll (2003, page 108). Because the estimation involves the definition of a linear regression model it can be augmented for the estimation of the additional regression parameters. Table 4 lists the parameter value estimates for the semiparametric model for the demographic variables. Note that the spline we are using in this case is a quadratic.

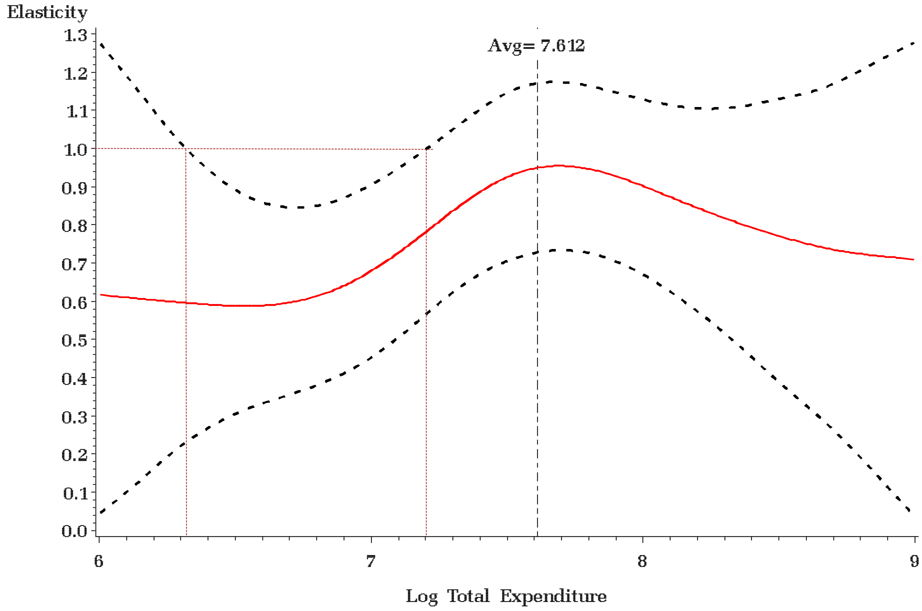
Table 4. The results of the semiparametric estimation as applied to the GVS data.

	Food		***Education***		***Alc & Tob***	
	Coefficient	***se***	***Coefficient***	***se***	***Coefficient***	***se***
Intercept	-180.16	136.59	16.45	11.79	24.32	20.86
AG	-58.25	22.57	2.35	2.85	1.77	3.09
AG2/100	66.53	26.10	-3.08	3.05	-5.02	3.58
DCOAST	-1.26	0.59	0.15	0.09	0.56	0.08
DMIDDLE	-6.61	0.52	0.03	0.08	0.23	0.07
CHILD6	-6.71	4.28	0.27	0.48	-1.24	0.59
GIRL6	0.18	3.93	-0.49	0.58	-0.01	0.54
*CHILD12/PUP12**	-5.87	3.27	0.67	0.28	-0.99	0.45
*GIRL12/PUG12**	-0.42	2.20	-0.12	0.35	-0.17	0.30
*CHILD15/PUP15**	-4.95	3.41	1.42	0.36	-0.60	0.47
*GIRL15/PUG15**	-8.73	3.21	-0.09	0.50	-1.02	0.44
*CHILD18/PUP18**	-8.32	3.36	2.43	0.48	-1.02	0.46
*GIRL18/PUG18**	0.45	3.23	-1.77	0.70	-0.22	0.44
PADU	-2.09	2.22			0.41	0.30
PFADU	-1.65	2.30			-0.69	0.31
NUM	-1.82	0.29	0.03	0.04	-0.13	0.04
PUP19			4.80	0.80		
PUG19			-3.87	1.22		
$\bar{R}^2$	.3671		.0241		.0355	

* In following GVS for the case of Education these variables are defined as the PUP version – the proportion of children in school of this gender and age instead of the proportion of all children in this age and gender group.

The elasticity for this case $\hat{\eta}_{y|c_j} = \frac{\hat{\rho}_j}{\hat{\theta}_j}$ is defined where $\hat{\theta}_j = \hat{\alpha}_0 + \sum_{k=1}^{K} \hat{\alpha}_k x_{ik} + \hat{h}(\ln(c_i))$ and $\hat{\rho}_j = \hat{\theta}_j + \left(\frac{\partial \hat{h}(\ln(c_i))}{\partial \ln(c)}\right)$. However the computation of the marginal impact of log of total expenditure $\left(\frac{\partial \hat{h}(\ln(c_i))}{\partial \ln(c)}\right)$ requires the computation of a numerical derivative evaluated at each level of the total expenditure. In this application we follow Wang and Wahba (1995) and use a model defined bootstrap where we first fit the semi-parametric model to the original sample and then we resample the residuals and add them back to the predicted values in order to create a new set of dependent variables as suggested by Freedman (1981). Thus we keep the independent variables the same and only change the dependent variables.

Figure 10 The elasticity of the expenditure share for alcohol and tobacco with respect to the log of total expenditure with 95% Fieller confidence intervals based on the semiparametric model.



From Figure 10 we can see that the elasticity estimates for alcohol and tobacco are significantly different from zero over the span plotted, however we can only reject unitary elasticity for the values of log expenditure from approximately 6.3 to 7.2.

Figure 11 The elasticity of the expenditure share for education with respect to the log of total expenditure with 95% Fieller confidence intervals based on the semiparametric model.

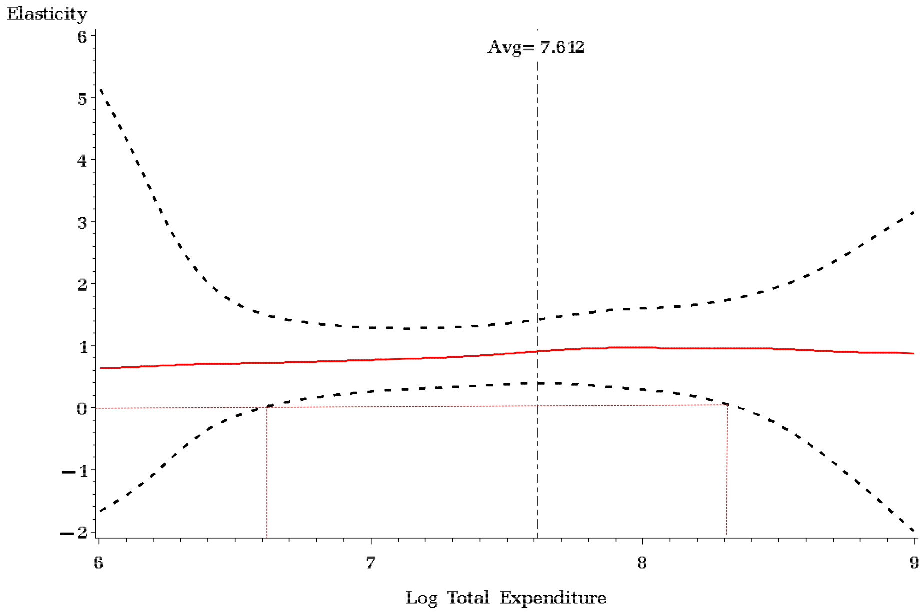


Figure 11 displays the expenditure elasticity for education as estimated by the semiparametric estimation. From this figure we find that we are unable to reject the hypothesis that the elasticity is equal to one for the entire span shown. However, we are only able to reject the hypothesis that these elasticities are zero for log total expenditures from 6.6 to 8.3.

Figure 12 The elasticity of the Expenditure share for food with respect to the log of total expenditure with 95% Fieller confidence intervals based on the semiparametric model.

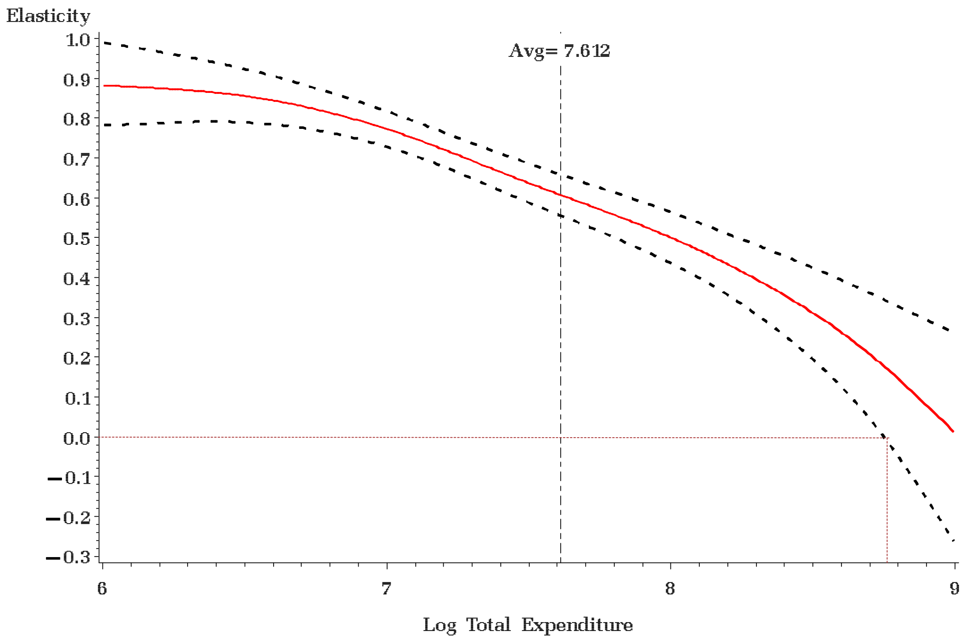


Figure 12 shows the plot of the elasticities for expenditure on food in which we note that for log total expenditures greater than 8.7 we cannot reject the hypothesis of a zero elasticity value. As we will show in Section 5 below due to the relatively tight fit of all the models for food the elasticities for food are very similar for all the estimation methods applied here.

5. Comparisons of Elasticity Estimates

Figures 14, 15 and 16 display the comparison plots of the elasticity and the 95% confidence upper and lower bounds by commodity and estimation strategy. In Figure 14 the large differences across models are most apparent at the lower levels of total expenditure. In particular, we find that the parametric elasticity estimates for alcohol and tobacco vary much more than those generated by the semiparametric model. For the education expenditures we also observe that, with the exception of the

Tobit model, the parametric models do not agree at lower levels of total expenditure. However, this must in a large part be a consequence of large proportion of the households with log total expenditure less than the mean (7.6) as having zero education expenditure. Interestingly, the penalized spline tracks the parametric Tobit model quite closely. All models for food consumption match each other quite closely which should not be surprising since food consumption is so well predicted by all the models.

Figure 14 A comparison of the estimated elasticities by commodity and estimation method.

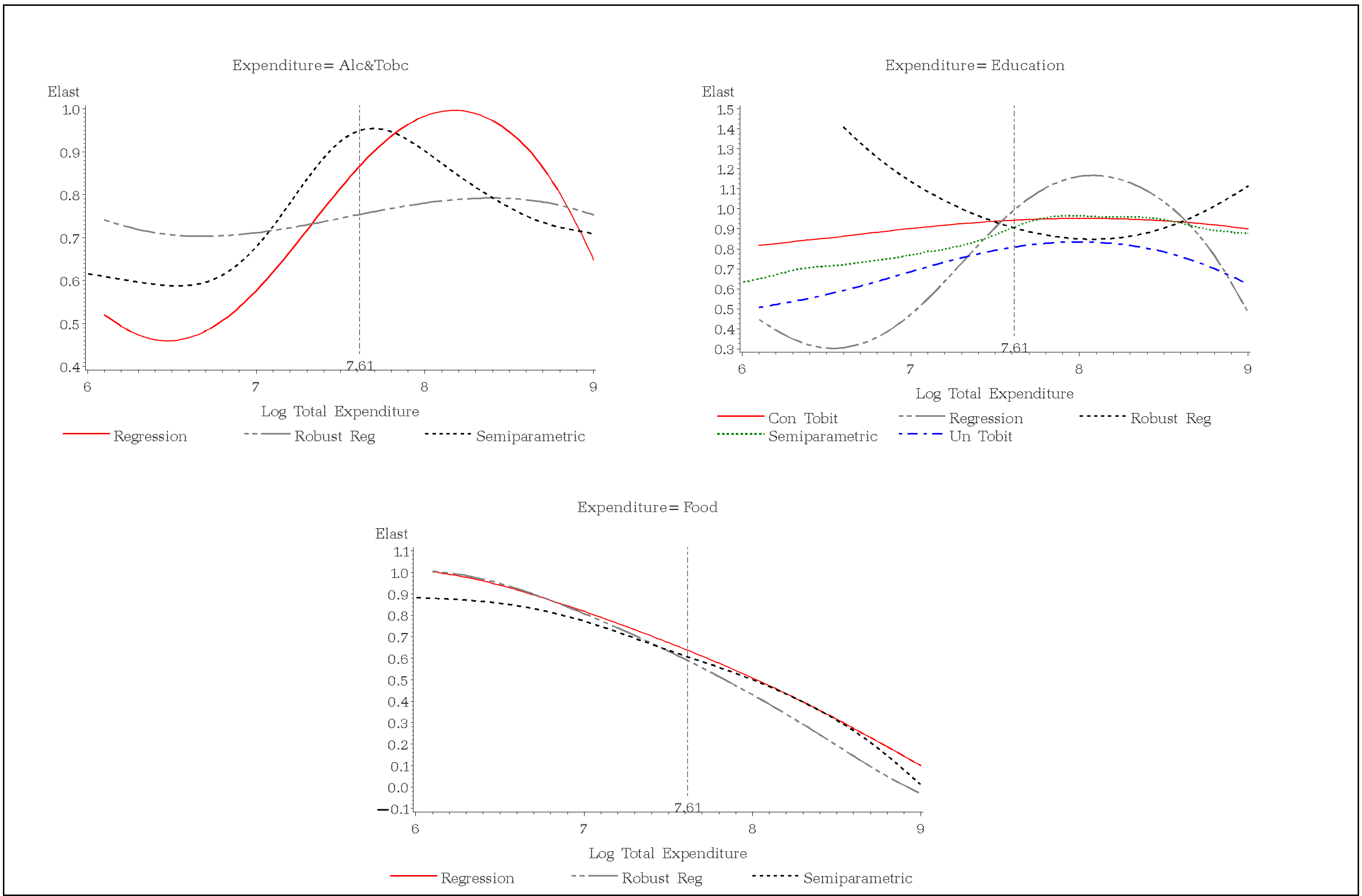


Figure 15 A comparison of the estimated Fieller upper 97.5% bound for elasticities by commodity and estimation method.

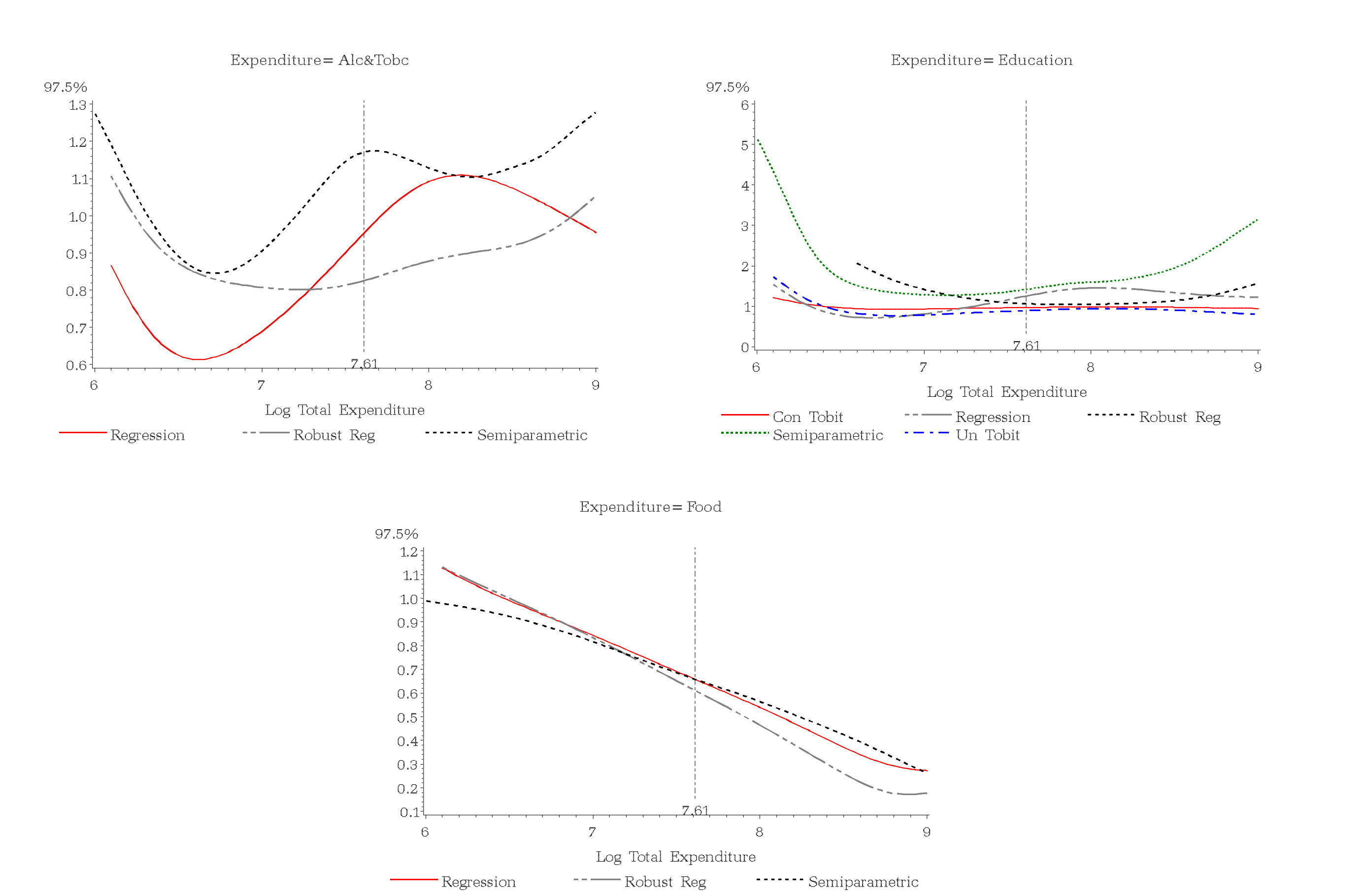
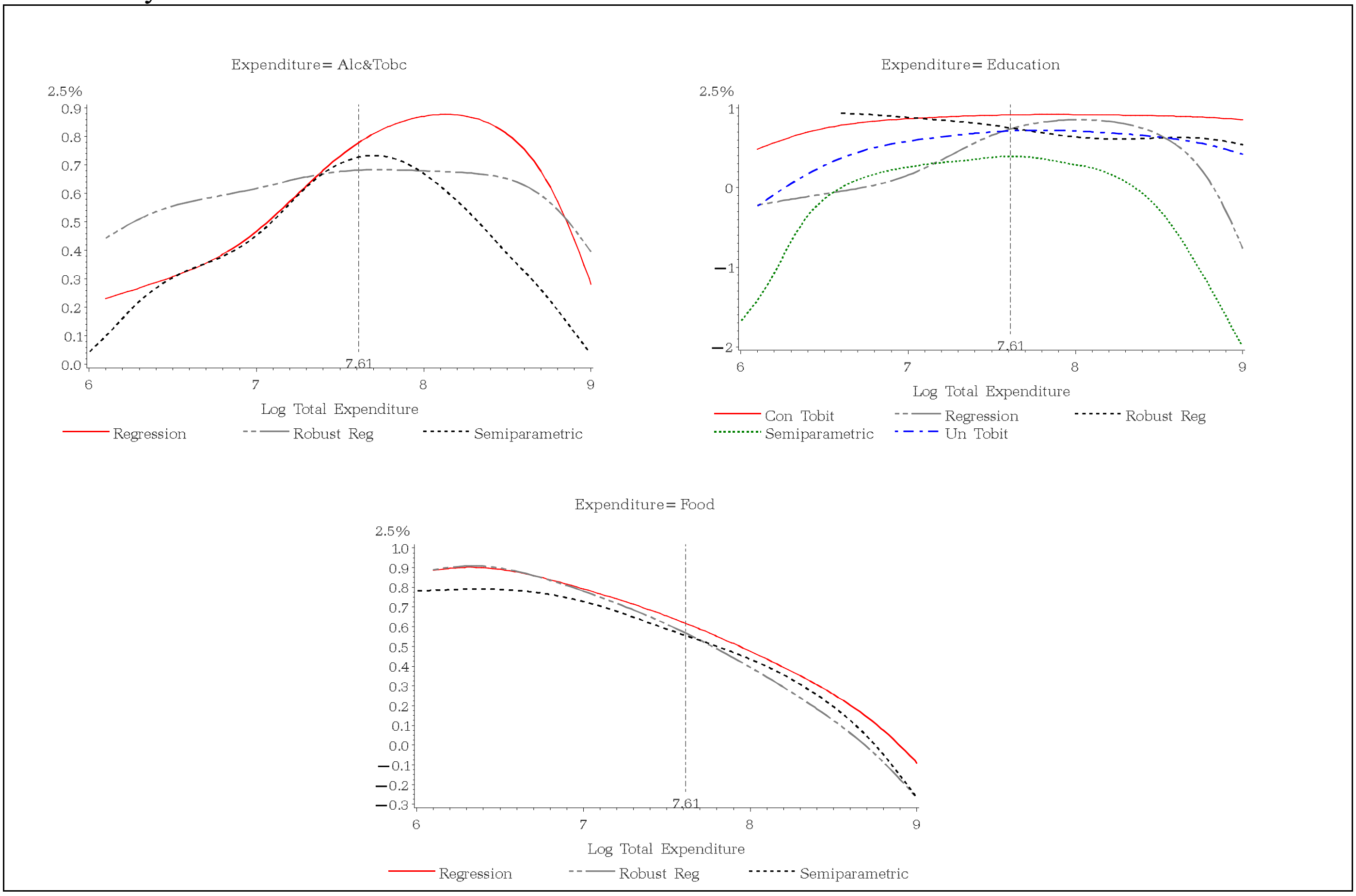


Figure 15 also shows a series of comparison plots for the estimated upper 97.5% bound based on the Fieller method. For alcohol and tobacco expenditure there is a uniform inference over the three models that the elasticity for log total expenditure from approximately 6.3 to 7.2 is less than one. In this case the robust model would indicate that for the majority of the values of log total expenditure the 97.5% upper bound of the elasticity is less than one. For education we find that all the models appear to indicate upper bounds greater than one for the majority of the sample. As with the elasticity estimates the upper bound values for food are fairly similar for the three estimation methods.

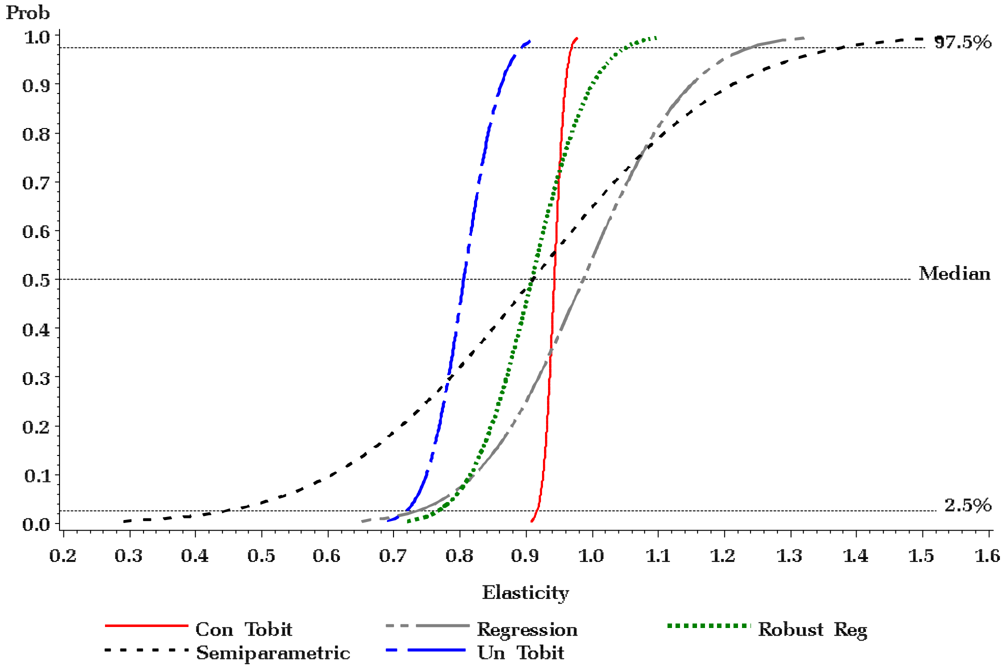
In Figure 16 we have plotted a series of the estimated 2.5 % bounds for the elasticities based on the Fieller method. From plots of the lower bound plots we can infer that the three methods used to model Alcohol and Tobacco expenditures result in elasticities that are greater than zero. For education this is true for all methods for the majority of the levels of the total expenditure in the sample. In the case of food it is only when approaching the highest value of the log total expenditure do we find values that are not significantly above zero.

Figure 16 A comparison of the estimated lower Fieller bound for a 2.5% bound for elasticities by commodity and estimation method.



In addition to the comparison of the elasticity measures we can also compare the precision of the elasticity measures across different levels of the log total expenditure and across estimation methods. In Figure 17 we compare the CDFs based on the Fieller method for education at the mean level of the log total expenditure as 7.6. Note that the expected value of the elasticity is marked for each model estimate. From these CDFs one can determine for each method how the elasticity varies and how the various probability statements one can make about the elasticities will vary by model used for estimation. The slopes of the CDFs indicate the precision of the estimates and the locations of the expected value of the elasticity (where p=.5). Thus we find that the steepest CDF is for the Tobit model and the least precise the elasticity estimate from the semiparametric model. Another observation from Figure 17 is the coincidence of the 2.5% lower bound for the three parametric estimates of the elasticity – they all have lower bounds around .7. However their 97.5% upper bounds appear to vary markedly from .9 for the Tobit to 1.3 for OLS.

Figure 17 The CDFs based on the Fieller method for the elasticity of expenditure on education when log total expenditure = 7.61.



An alternative comparison can be made for a particular method and commodity across different values of the log expenditure function in order to establish how the confidence intervals vary by value at which they are evaluated. In Figure 18 we plot the CDFs for the expenditure elasticity for food based on the results for the semiparametric model using the Fieller confidence interval method. As the level of expenditures decline we find that the expected value of the elasticities decline as well. We can also see from this diagram that the confidence intervals are fairly similar in size for log expenditure levels of 7, 7.6 and 8. However for log expenditures of 6 and 9 we see that the CDFs are markedly flatter indicating a widening of the confidence intervals.

Figure 18 The CDFs based on the Fieller method for the elasticity of expenditure on food based on the results from the semiparametric estimation.

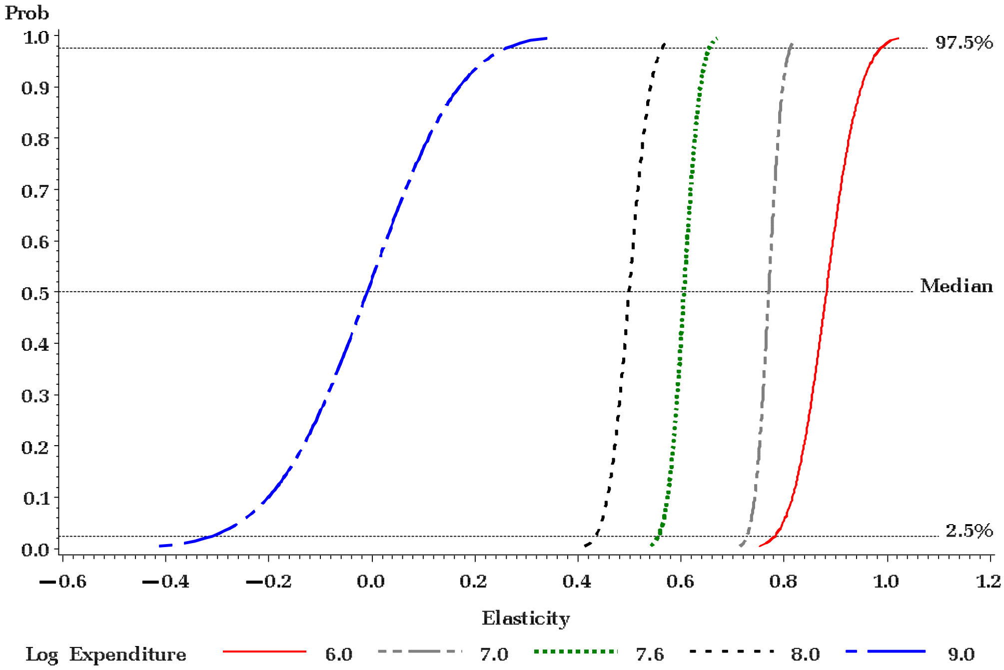
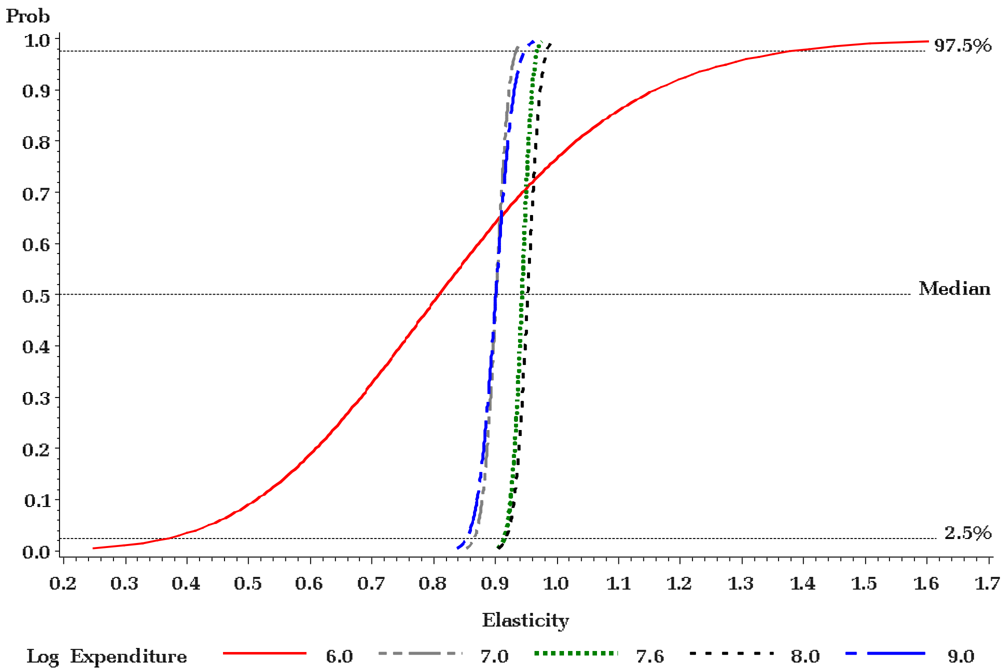


Figure 19 is the equivalent plot to Figure 18 for education expenses when using the Tobit model. In this case the CDF for the elasticity for an income at which almost no households consume educational services is shown to be much less precise than the case compared to the higher total expenditure levels.

Figure 19 The CDFs based on the Fieller method for the unconditional elasticity of expenditure on education based on the results from the conditional Tobit model estimation.



6. Conclusions

In this paper we have demonstrated that the Fieller intervals for the elasticity estimates can be implemented with a number of different estimation strategies. When the estimated parameters that make up the ratio that defines the elasticity are normally distributed the Fieller provides the exact confidence interval but the Delta method is only an approximation. When the estimated parameters are asymptotically normally distributed the Fieller is an approximation while the Delta method becomes an approximation based on an approximation. In this paper we have shown under what conditions the Fieller and the Delta are similar and what factors in the joint distribution of the estimates of numerator and denominator will lead to the two methods resulting in divergent inferences. A geometric examination of the relationship between these two methods is available in Hirschberg and Lye (2008).

In our application we find that the different models used to estimate the Engel curves for the same commodities do not result in the same point estimates of the elasticities. However, the inferences drawn as to whether the commodity has an income elasticity greater than one or not are quite similar across all values of total expenditure when we use the appropriately defined confidence intervals. We have also demonstrated that plots of the estimated elasticity CDF may be useful for the determination of the appropriate inferences especially in the cases where bounds may become infinite due to the available evidence

There are a number of alternative methods for the estimation of Engel curves that we have not considered. Alternative censored and semiparametric regression models have been proposed that will influence the form of the specific formulas used for the estimation. In addition, to other single equation robust methods quantile regression methods have also been applied to the estimation of Engel curves. It may also be possible to use methods other than the bootstrap for the estimation of the variance covariance matrix of the numerator and denominator for the elasticities from these estimation procedures. Our use of the bootstrap is limited in that we do not use the bootstrap to estimate the distribution of the ratios directly. Our main reason for this is that traditional resampling techniques applied to the ratio of means (see Davison and Hinkley (1997) for an extensive set of examples) the bounds are finite in nature and they do not allow for the open ended interval case. The specification of the constrained optimization in (7) implies that it is possible to construct Fieller-like intervals that allows for the use of assumptions for the joint distribution of the numerator and denominator other than the normal. Hirschberg and Lye (2007b) propose the use of an empirical joint distribution based on the bootstrap for the case of cost-effectiveness ratios.

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Appendix

The description of the data used from the Gong, Van Soest and Zhang (2005).

Table A.1 Summary Statistics for the 5,394 observations used in this analysis.

Variable name	*Label*	*Mean*	*StdDev*	*Min*	*Max*
FS	Food Expense Share x 100	54.786	18.656	3.300	99.606
AT	Alcohol & Tobacco Share x 100	1.850	2.071	0.000	34.165
ED	Education Share x 100	0.697	2.212	0.000	60.811
AG	Age divided by 100	0.418	0.094	0.220	0.835
DCOAST	dummy, 1 if household in coastal area	0.317	0.465	0.000	1.000
DMIDDLE	dummy, 1 if household in middle area	0.454	0.498	0.000	1.000
CHILD6	Proportion of children (0-5)	0.043	0.109	0.000	0.600
GIRL6	Proportion of female children (0-5)	0.019	0.071	0.000	0.500
CHILD12	Proportion of children (6-12)	0.155	0.193	0.000	0.667
GIRL12	Proportion of female children(6-12)	0.071	0.128	0.000	0.667
CHILD15	Proportion of children (6-12)}	0.070	0.120	0.000	0.600
GIRL15	Proportion of female children (13-15)	0.031	0.082	0.000	0.500
CHILD18	Proportion of children (16-18)	0.069	0.121	0.000	0.600
GIRL18	Proportion of female children (16-18)	0.032	0.084	0.000	0.500
PADU	Proportion of adult members (19+) (both parents and children older than 18)	0.603	0.213	0.000	1.000
PFADU	Proportion of female adult members (19+) (spouse and female children older than 18)	0.298	0.131	0.000	0.833
PUP12	Proportion of children at school (6-12)	0.131	0.180	0.000	0.667
PUG12	Proportion of female children at school (6-12)	0.059	0.117	0.000	0.600
PUP15	Proportion of children at school (13-15)	0.061	0.114	0.000	0.600
PUG15	Proportion of female children at school (13-15)	0.026	0.076	0.000	0.500
PUP18	Proportion of children at school (16-18)	0.032	0.087	0.000	0.600
PUG18	Proportion of female children at school (16-18)	0.014	0.057	0.000	0.500
PUP19	Proportion of children at school (19+)	0.010	0.050	0.000	0.600
PUG19	Proportion of female children at school (19+)	0.004	0.032	0.000	0.400
NUM	number of household members	3.966	0.977	2.000	9.000
LNCPER	log total expenditures per capita (yuan)	7.612	0.584	5.243	9.868
LNCPER2	*LNCPER* squared	58.287	9.036	27.485	97.370
ILNCPER	inverse *LNCPER*	0.132	0.010	0.101	0.191
ILNCPER2	inverse *LNCPER* squared	0.018	0.003	0.010	0.036

Figure A.1 The kernel density estimate for the log of the total household expenditure