Finding the invisible hand: an objective model of financial markets

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Abstract

A learning model is presented which resolves the ambiguity in the efficient markets concept because it is consistent with rational expectations. Heterogeneous least squares learning alone is sufficient to construct a price which contains all information. Contrary to intuition, there is no need for costly fundamental analysis; instead price is the product of interlocking expectations, and the continual revision of expectations causes price to gravitate to the efficient point. Price bubbles are not an aberration but an intrinsic part of a two phase process (normal / bubble) driven by the proportion of investors who consider price in their decisions. This behaviour can be understood in terms of a hidden substrate in which price is an information storing object, analogous to a set of genes in biology.

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JEL classification: G14

1. How do financial markets work – the efficient markets hypothesis

The Grossman Stiglitz (1980) paradox can be expressed more simply by moving it from a microeconomic framework to the language of rational expectations. How can the positive profit expectations of investors be reconciled with outcomes which have a net realization across buyers and sellers of zero? This question is particularly challenging in a single period model where the zero sum nature of the game is starkly apparent. There is no positive risk-free return nor any equity risk premium to be shared amongst the participants. This paper purports to construct an answer which applies even in this case. In short the answer is: the use of price as a variable in estimating return is costly most of the time because of the imprecise nature of price compared with other variables. Nonetheless at least some investors will continue to use price as a variable, and subsidize other investors in normal times, because it is profitable to use price in bubble/crash situations. Bubble/crashes are not an aberration but an intrinsic part of the two-phase cyclical process (stable / bubble) which governs financial markets.

1.2. The microeconomic examination of the efficient markets hypothesis

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The Grossman Stiglitz (1980) study used the following specific elements which have characterized microeconomic studies of financial markets ever since:

- There are two classes of trader, fundamental analysts and price watchers. The fundamental analysts value the stock using non-price information, which is represented by a scalar variable in the model.
- Fundamental analysis incurs cost, whereas price watching does not.
- Price watchers learn from price according to the standard rational expectations assumption that they have full knowledge of the return distribution conditioned on price.
- There is 'strategy switching', which means that traders can move between fundamental analysis and price watching according to relative profitability of either strategy.
- trader demand derives from explicit risk preference functions in a universe which contains a risk free security.

Grossman Stiglitz found that in a non stochastic environment price is perfectly revealing of the information of the fundamental analysts and no equilibrium can exist. If anyone is paying for information then everybody gets it, but if no one is paying for information then everyone wants it. They conclude (1980, pg 405), "because information is costly, prices cannot perfectly reflect the information which is available, since if it did, those who spent resources to obtain it would receive no compensation". Grossman Stiglitz employ a further element in their model:

- An exogenous source of noise which prevents price being fully revealing. Noise can be inserted in various ways to much the same effect; the particular device employed by Grossman Stiglitz is variable security supply.

The Grossman Stiglitz model is formulated in terms of conditional probability distributions. This version of the rational expectations hypothesis has been criticized for requiring actors to have information which is implausibly good. Investors cannot directly observe this information, nor can we observe that investors have it. Where actors are heterogeneous –have different information - there is the logical problem of assuming that each actor has expectations which are determined in part by their knowledge of the expectations of other actors using different information. Therefore Bray (1982) made an important contribution when she looked at price watchers using least squares learning to determine the relationship between price and return instead of rational expectations omniscience. Least squares learning refers to the econometric process by which agents regress observed return on price using ordinary least squares regression (OLS). They then use the estimated regression coefficients to derive predicted return for subsequent periods, and this predicted return is an argument of their demand function. Bray finds that the system will still converge to the rational expectations equilibrium with minimal conditions. Routledge (1999) extended this by specifying a model in which price watchers use least squares learning to learn the relationship between price and return, and all agents use adaptive learning to discern the best strategy. Adaptive learning is a different type of learning to least squares learning; instead of observing market data the agent observes the strategies and outcomes of other agents. It is assumed that agents move over time to the most successful strategies. Again the system still converges to the rational expectations equilibrium providing ‘experimentation noise’ in the strategy selection is low relative to exogenous noise.

The role of the noise trader in financial market economics is ambiguous. Noise traders have expectations which do not reflect market fundamentals: the precise implementation of the concept varies from paper to paper. Noise traders are normally regarded as violating the rational expectations paradigm because as argued in Friedman (1953) their behaviour will be unprofitable. For Friedman, noise traders cannot exist except in a transitory and inconsequential way. Black (1986) agrees that noise traders trade unprofitably, but argues that notwithstanding it is an observable fact that they do exist and have an important economic role. Their losses subsidize other market participants and underwrite the viability of fundamental analysis. Some take the argument even further and claim that noise trading can be profitable. Long Schleifer Summers and Waldmann (1990) construct a model in which noise traders can make money in the long term by taking on a large amount of risk. Rational investors do not arbitrage the noise traders to the true value as per the Friedman argument because it is too risky to do so; they fear the unpredictability of the noise trader’s behaviour and its attendant effect on price. Interestingly, Chiarella Gallegati Leombruni and Palestrini (2003) get a similar result using a model with fundamental traders and price watchers. Although the mathematical implementation differs substantially to the LSSW study, the conclusion that price watchers can make money by exploiting the price dynamics is the same. This is quite different to the Grossman Stiglitz notion that price watchers make money by tagging along after the fundamental analysts.

These last two studies are only two from the class of dynamic models. As Chiarella and He (2002) state, the development of mathematically sophisticated dynamic models has challenged the efficient market hypothesis from a different direction by demonstrating the potentially wide variety of price behaviour which markets might generate endogenously as different types of investor interact. Such behaviour can be construed as demonstrating that prices do not reflect fundamentals only. The influential paper Brock and Hommes (1997) generalizes the way in which cost is applied within the microeconomic model. Within the Brock Hommes model, simple estimators such as adaptive expectations are free, and rational expectations estimators are costly. It is assumed that if everyone uses the simple estimators, the equilibrium will be unstable. The price in the model oscillates as the effective estimators only pay for themselves when the price is somewhat inaccurate. The paper established an attractive system for chaotic modeling – in the words of Chiarella and He (2001 pg 501), “The resulting dynamical system is capable of generating the entire ‘zoo’ of complex behaviour from local stability to high order cycles and chaos as various key parameters of the model change.” Brock Hommes (1998) deals more explicitly with fundamental analysts and price chasers as well as agents using rational expectations. Further extensions have come from Chiarella and He (2001, 2002, 2003), and Branch and Evans (2006). In this last it is not cost differentials which motivate choice of estimator, but uncertainty as to the correct econometric specification. This class of financial model works with stylized properties of estimators rather than the estimators themselves, and stylized properties of different investor types. The nature of the dynamics can depend critically on parameter values. Nonetheless the models are suggestive of economic processes more complex and perhaps more interesting than those of the efficient markets hypothesis. Where strategy switching mechanisms are in place, they show that most of the time the different types of market player can all exist in a dynamic equilibrium.

It does not automatically follow that endogenous price movements are contrary to market efficiency if efficiency is interpreted in probabilistic terms: the expected value of price is correct and the variations are not great. Goldbaum in a series of papers (2005, 2006, 2007) finds that a non-stationary dividend process together with simultaneous least squares learning and strategy selection processes will generate endogenous noise. He regards endogenous noise as an intrinsic part of the economic process which serves to maintain the viability of fundamental analysis. The current paper develops this perspective by taking a heterogeneous least squares learning approach to fundamental analysis. We find that fundamental analysis will itself generate endogenous noise without the need for particular assumptions as to the dividend process or even strategy selection.

1.3. The point of departure of the objective model

It is argued here that a fuller understanding of price and investor price analysis is the key to a consistent and realistic model of financial markets. Security return is considered in multivariate generality rather than as the sum of one or two components. We take a heterogeneous least squares learning approach to fundamental analysis: every agent uses least squares learning and different agents use different data sets. It might be said that:

- all agents are fundamental analysts, because they may (or may not) look at more data than price alone.
- no agents are fundamental analysts, because all of them use least squares learning rather than additive approaches to determine the value of the security. ‘Additive’ here means a method which purports to arrive at a value for a security using all available information processed according to accounting first principles.

Essentially then, the model reinterprets the concepts of fundamental analysis and the dichotomy of fundamental/price analysis which have been the mainstay of the microeconomic literature on financial markets.

If cost differentials were eliminated in traditional analyst/price taker models then those models would favour analysts and resolve the paradox. This is not a natural thing to do because those models rely on cost for economic context and would become trite in its absence. Within this model a multivariate treatment of return and heterogeneous information is the primary source of economic context. Cost determines the position of the equilibrium but cannot affect the nature of the equilibrium. This is consistent with some of the more recent literature (Goldbaum (2005), Branch and Evans (2006)) which also deemphasizes cost. Strategy switching works with cost to determine the position of equilibrium, but strategy switching is not emphasized here because the focus of the objective model is on short term processes. Risk and portfolio considerations are likewise kept in the background. The demand and supply equations derive from a mean-variance framework.

The contribution of the paper is as follows. A model is developed which explores multivariate heterogeneous least squares learning within the context of rational learning, a learning version of rational expectations. The model is set in discrete time. It is an analytic model using constructive mathematical proofs as distinct from proofs of existence. Simulations are used at some points to illustrate the points being made and validate results which depend on first order approximations. Section 2 presents the objective model. Price is shown to be a linear function of market data, and this allows data to be eliminated from the model in order to reveal the underlying structure. Section 3 develops the price change equation, which states that price changes are not in the direction of the change of estimates *new minus old* as might be expected, but in the direction of *new* estimates. This is because both price and data coefficients are estimated, and the *old* components cancel out. Price changes push price in the direction of the unknown return parameters so the market is efficient. Section 4 undertakes the technical task of demonstrating that that the price regression coefficient is negative in a model where it is derived from regression rather than prior expectation. A negative value of the price coefficient is required for the market to make a price. Section 5 investigates the economics of the model. Contrary to the Grossman Stiglitz paradox, investors are rewarded according to the value of their information although this appears to imply a two phase process in which markets cycle between normal and bubble behaviour. Section 6 reviews major themes. The flip side of the requirements for market efficiency are the reasons for market bubbles and this is discussed. The genetic analogy is not incidental but relevant to the paper's central conclusion – that price is not an immediate function of other variables but a store of information with an independent existence, like a gene. It can be regarded as an object in the sense that term is used in computing - an independent entity with well defined properties and methods. For this reason the model is referred to as the objective model, i.e. objective in the sense of 'referring to objects', not in the sense of 'independently verifiable as opposed to subjective'.

2. Definition of the objective model

2.1. Premises of the objective model

PREMISE 1: RATIONAL LEARNING. Rational expectations is not interpreted in the 'omniscient' way whereby actors have perfect knowledge of underlying probability distributions conditioned by the information they possess. Rather we recognize that forming expectations under heterogeneity is complex and actors must use an adaptive stepwise technique. Expectations formed through the learning process must be realized to some degree or that kind of learning will not be sustained – actors 'learn about learning'. It is this which distinguishes rational learning from the naïve adaptive expectations concept which rational expectations replaced. It is of course possible that the actors in the model will step to the wrong place - a local optimum rather than a global one - but that would seem to be as much a feature of the real world as the modeling technique, and one of the objects of investigation is to determine whether such an outcome is possible.

PREMISE 2: BASIC FRAMEWORK. There is a security. Payments at the rate of y per unit are made to security holders at fixed points of time spaced out at equal intervals, for instance every midday. The payment may be negative: for instance the security may be insurance policies. At the start of each period, investors indicate their interest by bidding for the stock. There is zero net supply, investors can long or short the stock and the Walrasian auctioneer sets the market clearing price.

The period between one payment and the next is referred to as the 'observation' period. In the one period model, the security is extinguished once the payment is made so the payment must include any return of capital. Another security of identical characteristics comes into being after the payment is made. In the multiperiod model, the payment is a dividend and the security continues on.

The market for the security consists of J investors each of whom derive their own model of security return by carrying out OLS regression on observed security returns using market price and variables of their own choosing. Each investor uses the regression coefficients which he or she has estimated to predict future returns.

The sequence of observation periods is divided up into longer segments, each T observation periods long, which are referred to as 'estimation' periods. The estimation periods are the same for every investor. At the end of each estimation period, a *small proportion* of investors re-estimate their model using the T observations made in the period. They replace their old coefficient estimates with the fresh estimates and use them henceforth. The other investors keep using the coefficient estimates they already have. The only time coefficient estimates are updated is at the end of each estimation period; for all observation periods within a particular estimation period the coefficient estimates are kept the same.

PREMISE 3: DATA SET. Investors use different data, but the data set $\mathbf{X}_{T*N\,orig}^{original}$ (covering T observation periods for a set of $N^{original}$ variables) is a compilation which lists every variable used by every investor, probably with repetition. Also available is the set of absolute returns for the security $\mathbf{y}_{T*1}$ which is the same for every investor. Each item of data is available at the start of the observation period and the corresponding return is measured at the end of the observation period.

The initial data set $\mathbf{X}^{original}$ has a basis $\mathbf{X}$ containing N vectors, $N \le T$. It is supposed that the basis is chosen so that the variance of the basis variables is normalized orthogonal:

$$E\left[\mathbf{X}'\mathbf{X}\right] = \mathbf{I}_{N*N} \tag{1}$$

and it is also supposed with some degree of approximation that the basis is the same in every period and displays exact orthogonality:

$$\mathbf{X}_1 = \mathbf{X}_0 \tag{2}$$

$$\mathbf{X'X} = \mathbf{I}_{N*N} \tag{3}$$

PREMISE 4: INVESTOR STRATEGIES. The specific information which an investor uses is referred to as their strategy. Each investor *j* is assumed to employ *jK* different variables in their attempt to explain **y**.

$$0 \le jK \le N \tag{4}$$

Given the data transformation specified above, the variables are not confined to the exact variables in **X** but may include linear functions of these variables. Each variable $\mathbf{X}_{jk}$ which investor *j* uses is derived from the data set using an $N*1$ strategy vector $\mathbf{a}_{jk}$ via:

$$\underset{T*1}{\mathbf{X}_{jk}} = \underset{T*N}{\mathbf{X}} \cdot \underset{N*1}{\mathbf{a}_{jk}} \tag{5}$$

These strategy vectors are fixed. The $\mathbf{a}_{jk}$ vectors can be assembled into one strategy matrix $\mathbf{a}_j = \begin{bmatrix} \mathbf{a}_{j1} & & \mathbf{a}_{jK} \end{bmatrix}$ so that the full data set $\mathbf{X}_j$ used by the investor *j* is given by:

$$\underset{T*jK}{\mathbf{X}_j} = \mathbf{X}_{T*N} \cdot \underset{N*jK}{\mathbf{a}_j} \tag{6}$$

We assume that each investor uses independent variables in their regressions, so that the strategy matrix $\mathbf{a}_j$ is of full rank:

$$rank\left(\mathbf{a}_j\right) = jK \le N \tag{7}$$

Putting together the $\mathbf{a}_j$ matrices for every investor to give one matrix **a** gives:

$$\mathbf{X}\mathbf{a}_{N*sum(jK)} = \mathbf{X}^{original} \qquad \text{where of course} \quad sum(jK) = N^{orig} \tag{8}$$

Given that $rank(\mathbf{X}^{original}) = rank(\mathbf{X}) = N$ it follows that:

$$rank(\mathbf{a}) = N \tag{9}$$

or in other words, the set of strategy vectors $\left\{\mathbf{a}_{jk}\right\}$ span the space $\mathbb{R}^N$.

An investor may use all the available data but not price (a fundamentalist) or use price only (a price watcher) or use some data together with or without price. In this model the fundamentalist / price watcher categories represent the two ends of a continuum of investor behaviour.

PREMISE 5: PARTICULAR PROPERTIES OF PRICE. In addition to the data in data set **X**, at least one investor includes the price of the security $\mathbf{p}_{T*1}$ as a variable in their data set $\mathbf{X}_j$. Price is not included as part of another variable $\mathbf{X}_{jk}$ but only in its own right.

PREMISE 6: GENERATION OF RETURNS. It is assumed that the return per period per unit (one share, not one dollar's worth) of the security is generated according to:

$$\mathbf{y} = \mathbf{X}\boldsymbol{\mu} + \mathbf{u} \quad \text{where } \mathbf{X'X} = \mathbf{I}_{N*N}, \quad \mathrm{E}(\mathbf{u}) = 0, \quad \mathrm{E}\left[\mathbf{uu'}\right] = \sigma^2 \mathbf{I}_{T*T} \tag{10}$$

The total variance $\mathrm{E}\left[\mathbf{u'u}\right] = \sigma^2 \boldsymbol{\iota}'_{1*N} \cdot \mathbf{I}_{N*N} \boldsymbol{\iota}_{N*1} = N\sigma^2$ is denoted σ^2_{tot}.

The assumption that the mean is zero is not restrictive as OLS will automatically generate this result when an intercept is included (although it has not been assumed that the constant vector $\boldsymbol{\iota}$ occurs in **X**) . This relation is an empirical regularity for the particular data set **X** which does not purport to be causative; if investors had a different data set available to them then a different relation would apply. Nonetheless we suppose that the return vector $\boldsymbol{\mu}$ has the character of a parameter in that it is stable over time.

Although the variance of the dividend process is finite this does not rule out an infinite variance for security return (which includes price fluctuations) in the multiperiod model.

PREMISE 7: INDEPENDENT RE-ESTIMATION. As stated, at the end of a particular estimation period a small minority of investors take the opportunity to upgrade their estimates using **X.** These investors are referred to as '*new* investors', although it is the estimates which are new and not the investors themselves. The majority of investors ('*old* investors') continue to use the estimates they already have. The proportion of this minority to the whole is drawn randomly from the ranks of the investors. Thus some investors may re-estimate immediately, others may not re-estimate for a long time. The point of this independence assumption is that the average coefficient estimates of the *old* investors remains the same into the next estimation period. The process can also be interpreted as an geometric adaptive expectations process.

PREMISE 8: DEMAND PROPORTIONAL TO EXPECTED RETURN. The standard risk-theoretic development (given in Grossman 1976 pp 574-576) results in demand functions which are linear in expected return. In my notation:

$$q_j = B_j \hat{r}_j \tag{11}$$

where q_j is the amount of shares (units) in the security demanded by investor j, $\hat{r}_j$ is the net return which the investor predicts they will receive on each share of the security, and B_j is the amount of stock demanded by investor j per unit of net return: the "weight of money" brought to bear by investor j. It is always positive. It is helpful to define b_j , the relative weight of money for investor j, by

$$b_j = \frac{B_j}{\sum_j B_j} \tag{12}$$

We see immediately that:

$$b_j > 0 \quad \text{given } B_j > 0 \text{ for all investors.} \tag{13}$$

$$\sum_j b_j = 1 \tag{14}$$

Because there is no constant, a non zero quantity q_j requires that expectation $\hat{r}_j$ be non-zero, so any active investor must analyze at least one data series to generate a non-zero expectation.

Investors will find that return expectations are not exactly realized so it is a question as to whether demand equation (11) is consistent with rational learning. This question is revisited in Section 5.

2.2. Price theorem

RESULT 1. *The weighted predicted return is zero.*

$$\sum_j b_j \hat{r}_j = 0 \tag{15}$$

PROOF: $0 = \sum_j q_j$ by Premise 2 (16)

$= \sum_j B_j \hat{r}_j$ by (11) (17)

$= \sum_j b_j \hat{r}_j$ multiplying by $\frac{1}{\sum_j B_j}$ which is positive # (18)

Strictly speaking, this should be developed through a formal assumption that some of the information in the data set $\mathbf{X}$ is relevant (i.e. $\boldsymbol{\mu} \neq 0$), and consequently it is possible for some investor to form a non-zero expectation of return and cause price to be non-zero. This development is omitted for brevity.

The regression coefficients for investor *j* are denoted:

$$\hat{\boldsymbol{\beta}}_j = \begin{bmatrix} \hat{\boldsymbol{\beta}}_{jX} \\ \hat{\rho}_j \end{bmatrix} \tag{19}$$

where $\hat{\boldsymbol{\beta}}_{jX}$ denotes the vector of data regression coefficients and $\hat{\rho}_j$ denotes the price regression coefficient. $\hat{\mathbf{r}}_j$ denotes the T*1 expected return vector of investor *j* for the estimation period. So

$$\hat{\mathbf{r}}_j = \begin{bmatrix} \mathbf{X}_j & \mathbf{p} \end{bmatrix} \begin{bmatrix} \hat{\boldsymbol{\beta}}_{jX} \\ \hat{\rho}_j \end{bmatrix} \tag{20}$$

In the case of both (19) and (20) any particular investor *j* may use data only, price only, or both. They must use at least one series to generate a non-zero expectation.

RESULT 2: PRICE THEOREM. *Price can be expressed as a function* $\boldsymbol{\pi}$ *of* $\mathbf{X}$*, i.e.*

$$\underset{T*1}{\mathbf{p}} = \underset{T*N}{\mathbf{X}} \underset{N*1}{\boldsymbol{\pi}} \tag{21}$$

where $\boldsymbol{\pi} = -\frac{\sum_j b_j \mathbf{a}_j \hat{\boldsymbol{\beta}}_{jX}}{\rho}$ *is referred to as the price equation,* (22)

and $\rho = \sum_j b_j \hat{\rho}_j$ *is referred to as the price coefficient.* (23)

(Note the distinction between price coefficient ρ *and price regression coefficient* $\hat{\rho}_j$*.)*

PROOF: $\hat{\mathbf{r}}_j = \mathbf{X}\mathbf{a}_j \hat{\boldsymbol{\beta}}_{jX} + \mathbf{p}\hat{\rho}_j$ expanding (20) and using $\mathbf{X}_j = \mathbf{X}\mathbf{a}_j$ (24)

$\sum_j b_j \left(\mathbf{X}\mathbf{a}_j \hat{\boldsymbol{\beta}}_{jX} + \mathbf{p}\hat{\rho}_j \right) = 0$ substituting into (15) (25)

$$\mathbf{p} = -\frac{\mathbf{X}\cdot\sum_j b_j \mathbf{a}_j \hat{\boldsymbol{\beta}}_{jX}}{\sum_j b_j \hat{\rho}_j} \qquad \text{making } \mathbf{p} \text{ the subject} \qquad \# \qquad (26)$$

As trivial as this result appears it is of fundamental importance. If investors form expectations using a certain data set then price must be an exact linear function of that data set. There is no error term in equation (21). Because price has a stable relationship to the data then data does not matter in a sense and the problem can be abstracted to the coefficients of data.

This functional form is constant as long as the estimations in use are unchanged, i.e. throughout the estimation period. The result may suggest that if an investor has access to all the information **X**, they could estimate the price coefficient $\boldsymbol{\pi}$ perfectly. However there is no reason for an investor to do this; explaining current price is not the same thing as predicting return which depends on dividends and future price.

Expressions for expected and realized return: Realized return $\mathbf{r}_{T*1}$, the net absolute (not percentage) return, is given by the gross return less price paid.

$$\mathbf{r}_0 = \mathbf{X}_0\boldsymbol{\mu} + \mathbf{u}_0 - \mathbf{p}_0 \qquad \text{by (10)} \qquad (27)$$

$$= \mathbf{X}_0\boldsymbol{\mu} - \mathbf{X}_0\boldsymbol{\pi} + \mathbf{u}_0 \qquad \text{applying (21)} \qquad (28)$$

Define an augmented form of the strategy matrix $\mathbf{a}_j$ which includes price, denoted $\tilde{\mathbf{a}}_{j0}$.

$$\tilde{\mathbf{a}}_{j0} = \begin{bmatrix} \mathbf{a}_j & \boldsymbol{\pi}_0 \end{bmatrix} \qquad (29)$$

If investor *j* does not use price then $\tilde{\mathbf{a}}_j = \mathbf{a}_j$ and if investor *j* does not use non-price data then $\tilde{\mathbf{a}}_j = \boldsymbol{\pi}$. The number of columns in $\tilde{\mathbf{a}}_j$ is denoted jK^{aug}. If price is not included, $jK^{aug} = jK$; if price is included $jK^{aug} = jK + 1$. We take it that if the investor includes $\boldsymbol{\pi}$ in the strategy matrix, it is linearly independent of the other variables which they include.

$$rank\left(\tilde{\mathbf{a}}_j\right) = jK^{aug} \qquad (30)$$

The expression (20) for expected return $\hat{\mathbf{r}}_j$ can be restated in the following useful forms:

$$\hat{\mathbf{r}}_j = \begin{bmatrix} \mathbf{X}\mathbf{a}_j & \mathbf{X}\boldsymbol{\pi}_0 \end{bmatrix} \begin{bmatrix} \hat{\boldsymbol{\beta}}_{jX} \\ \hat{\rho}_j \end{bmatrix} \qquad \text{applying (6),(21) to (20)} \qquad (31)$$

$$= \mathbf{X}\begin{bmatrix} \mathbf{a}_j & \boldsymbol{\pi}_0 \end{bmatrix} \begin{bmatrix} \hat{\boldsymbol{\beta}}_{jX} \\ \hat{\rho}_j \end{bmatrix} \qquad \text{factorizing} \qquad (32)$$

$$= \mathbf{X}\tilde{\mathbf{a}}_{j0}\hat{\boldsymbol{\beta}}_j \qquad \text{by (29), (19)} \qquad (33)$$

3. The single period objective model

3.1. New coefficients

RESULT 3: ESTIMATION. *The estimates of the new investors obtained from data set* $\mathbf{X}_0$ *for estimation period 0 are given by:*

$$\hat{\boldsymbol{\beta}}_{j\,new} = \left(\tilde{\mathbf{a}}_{j0}{}'\tilde{\mathbf{a}}_{j0}\right)^{-1}\tilde{\mathbf{a}}_{j0}{}'\left(\boldsymbol{\mu} - \boldsymbol{\pi}_0 + \mathbf{e}_0\right) \quad \textit{where} \tag{34}$$

$$\mathbf{e}_{N*1} = \mathbf{X}_0{}'\mathbf{u}_0 = \left(\mathbf{X}_0{}'\mathbf{X}_0\right)^{-1}\mathbf{X}_0{}'\mathbf{u}_0, \quad \mathrm{E}\left[\mathbf{e}\right] = \mathbf{0}, \quad \mathrm{E}\left[\mathbf{e}\mathbf{e}'\right] = \sigma^2\mathbf{I}_{N*N} \tag{35}$$

PROOF: Assuming that investor *j* uses both **X** and **p** in their regressors, we get:

$$\begin{bmatrix}\hat{\boldsymbol{\beta}}_{j\,new}\\ \rho_{j\,new}\end{bmatrix} = \left(\begin{bmatrix}\mathbf{X}_{0j} & \mathbf{p}_0\end{bmatrix}'\begin{bmatrix}\mathbf{X}_{0j} & \mathbf{p}_0\end{bmatrix}\right)^{-1}\begin{bmatrix}\mathbf{X}_{0j} & \mathbf{p}_0\end{bmatrix}'\mathbf{r}_0 \quad \text{standard OLS formula} \tag{36}$$

$$= \left(\begin{bmatrix}\mathbf{a}_j & \boldsymbol{\pi}_0\end{bmatrix}'\mathbf{X}_0{}'\mathbf{X}_0\begin{bmatrix}\mathbf{a}_j & \boldsymbol{\pi}_0\end{bmatrix}\right)^{-1}\begin{bmatrix}\mathbf{a}_j & \boldsymbol{\pi}_0\end{bmatrix}'\mathbf{X}_0{}'\left(\mathbf{X}_0\boldsymbol{\mu} - \mathbf{X}_0\boldsymbol{\pi}_0 + \mathbf{u}_0\right) \quad \text{by(6),(28)} \tag{37}$$

$$= \left(\tilde{\mathbf{a}}_{j0}{}'\tilde{\mathbf{a}}_{j0}\right)^{-1}\tilde{\mathbf{a}}_{j0}{}'\left(\boldsymbol{\mu} - \boldsymbol{\pi}_0 + \mathbf{e}_0\right) \quad \text{by (3), (29), (35)} \tag{38}$$

Similarly if the investor omits **X** or **p** from their regressors.

Observe $\mathbf{e}_0 = \mathbf{X}_0{}'\mathbf{u}_0$, so $E\left[\mathbf{e}_0\mathbf{e}_0{}'\right] = E\left[\mathbf{X}_0{}'\mathbf{u}_0\mathbf{u}_0{}'\mathbf{X}_0\right] = \sigma^2\mathbf{I}_{N*N}$ # (39)

We see the interesting result that regression of return against data is equivalent to regression of the underlying parameters $\boldsymbol{\mu} - \boldsymbol{\pi}$ against $\tilde{\mathbf{a}}_{j0}$, the coefficients used by investor i to derive their variables. The investor's problem in a sense does not involve data but rather is a test of how well the investor's variable coefficients capture the return coefficients $\boldsymbol{\mu} - \boldsymbol{\pi}$. We have abstracted the problem from 'data space' $\mathbb{R}^T$ to 'coefficient space' $\mathbb{R}^N$; this is a substantial gain in simplicity.

3.2. Updating coefficients

We establish some obvious results about the relative weight of money b_j. Notation:

- *w* is the proportion of investors who reestimate in each period. By Premise 7, it is the same for every investor type *j*.
- b_j^t is the relative weight of money of those investors who last reestimated in period *t*.
- Period subscript denotes the value of the variable in that period. So b_{j1}^t denotes the value of b_j^t in period 1:

$$b_j = \sum_{t=1}^{-\infty} b_{j1}^t \tag{40}$$

$$b_{j1}^t = \left(1 - w\right)b_{j0}^t \quad \text{by Premise 7} \tag{41}$$

$$\hat{\boldsymbol{\beta}}_{j0} = \frac{\sum_{t=-1}^{-\infty} b_{j0}^t\hat{\boldsymbol{\beta}}_j^t}{b_j} \tag{42}$$

$$\hat{\rho}_{j0} = \frac{\sum_{t=-1}^{-\infty} b_{j0}^t \hat{\rho}_j^t}{b_j} \tag{43}$$

$$\hat{\rho}_0 = \frac{\sum_{j\,incl} b_j \hat{\rho}_{j0}}{\sum_{j\,incl} b_j} = \frac{\sum_{j\,incl} \sum_{t=-1}^{-\infty} b_{j0}^t \hat{\rho}_j^t}{\sum_{j\,incl} b_j} \tag{44}$$

where *j incl* refers only to those investor types *j* who include price.

- It is convenient to use *new* to denote investors who reestimated in period 0:

$$b_{j\,new} = b_{j1}^0 \tag{45}$$

RESULT 4: PROPORTION OF NEW INVESTORS.

$$b_{j\,new} = wb_j \tag{46}$$

PROOF: $$b_{j\,new} = b_j - \sum_{t=-1}^{-\infty} b_{j1}^t \quad \text{by (40)} \tag{47}$$

$$= b_j - \sum_{t=-1}^{-\infty} (1-w) b_{j0}^t \quad \text{by (41)} \tag{48}$$

$$= b_j - (1-w) \sum_{t=-1}^{-\infty} b_{j0}^t = b_j - (1-w) b_j = wb_j \quad \# \tag{49}$$

For the purposes of the next two results we introduce:

$$\hat{\boldsymbol{\beta}}_{j\,new} = \hat{\boldsymbol{\beta}}_j^0 \tag{50}$$

$$\hat{\rho}_{new} = \frac{\sum_{j\,incl} b_{j1}^0 \hat{\rho}_j^0}{\sum_{j\,incl} b_{j1}^0} = \frac{\sum_{j\,incl} b_j \hat{\rho}_j^0}{\sum_{j\,incl} b_j} \quad \text{by (46)} \tag{51}$$

RESULT 5: UPDATE LEMMA FOR REGRESSION COEFFICIENT.

$$\hat{\boldsymbol{\beta}}_{j1} = w\hat{\boldsymbol{\beta}}_{j\,new} + (1-w)\hat{\boldsymbol{\beta}}_{j0} \tag{52}$$

PROOF: $$\hat{\boldsymbol{\beta}}_{j1} = \frac{b_{j1}^0 \hat{\boldsymbol{\beta}}_j^0 + \sum_{t=-1}^{-\infty} b_{j1}^t \hat{\boldsymbol{\beta}}_j^t}{b_j} \quad \text{applying (42) to period 1} \tag{53}$$

$$= \frac{wb_j \hat{\boldsymbol{\beta}}_j^0 + (1-w) \sum_{t=-1}^{-\infty} b_{j0}^t \hat{\boldsymbol{\beta}}_j^t}{b_j} \quad \text{by (46),(41). Apply (50),(42) for result.\#} \tag{54}$$

In the case of the price coefficient there are many different investor types estimating the one coefficient. In this case the proof must be elaborated as per:

RESULT 6: UPDATE LEMMA FOR PRICE REGRESSION COEFFICIENT.

$$\hat{\rho}_1 = w\hat{\rho}_{new} + (1-w)\hat{\rho}_0 \tag{55}$$

PROOF: $\hat{\rho}_1 = \dfrac{\sum_{j\,incl} b_{j1}^0 \hat{\rho}_j^0 + \sum_{j\,incl} \sum_{t=-1}^{-\infty} b_{j1}^t \hat{\rho}_j^t}{\sum_{j\,incl} b_j}$ applying (44) to period 1 (56)

$$= \frac{w \sum_{j\,incl} b_j \hat{\rho}_j^0 + (1-w) \sum_{j\,incl} \sum_{t=-1}^{-\infty} b_{j0}^t \hat{\rho}_j^t}{\sum_{j\,incl} b_j} \quad \text{by (46),(41). Apply (51),(44) for result.\#} \quad (57)$$

The sum of the relative weight of money of all investors using price is denoted:

$$b_{price} = \sum_{j\,incl} b_j \quad (58)$$

RESULT 7: UPDATE LEMMA FOR PRICE COEFFICIENT.

$$\rho_1 = b_{price} (1-w) \hat{\rho}_0 + b_{price} w \cdot \hat{\rho}_{new} \quad (59)$$

PROOF: $\rho_1 = \sum_{j\,incl} b_j \hat{\rho}_{j1}$ by definition (23) (60)

$$= \hat{\rho}_1 \sum_{j\,incl} b_j = b_{price} \hat{\rho}_1 \quad \text{by (44),(58). Apply (55) for result. \#} \quad (61)$$

The price coefficient ρ is smaller than the price regression coefficient $\hat{\rho}$ because in general $b_{price} < 1$.

3.3. Price change theorem

We determine how the new return estimates derived in estimation period 0 affects the price in estimation period 1. We are now dealing with data set $\mathbf{X}_1$. The change in price $\mathbf{dp}$ is not measured relative to the prices $\mathbf{p}_0$ in estimation period 0 obtained with data $\mathbf{X}_0$, but relative to the price which would obtain with current data set $\mathbf{X}_1$ if there were no new estimates. Data set $\mathbf{X}_1$ is therefore taken as constant for purposes of differentiation.

RESULT 8: PRICE CHANGE THEOREM.

$$\mathbf{d\pi} = -\frac{dw}{\rho_0} \mathbf{H}_0 \left(\boldsymbol{\mu} - \boldsymbol{\pi}_0 + \mathbf{e}_0 \right) \quad (62)$$

where $\mathbf{H}_{N*N} = \sum_j b_j \tilde{\mathbf{a}}_{j0} \left(\tilde{\mathbf{a}}_{j0}' \tilde{\mathbf{a}}_{j0} \right)^{-1} \tilde{\mathbf{a}}_{j0}'$ (63)

is referred to as the "estimation matrix" and

$$\rho_0 = \sum_{j\,old} b_{j\,old} \hat{\rho}_{j\,old} \quad (64)$$

where subscript 0 refers to estimation period 0.

PROOF: $\mathbf{0}_{T*1} = \sum_j b_j \hat{\mathbf{r}}_j$ by (15) (65)

$$= \sum_j \sum_{t=0}^{-\infty} b_{j1}^t \left(\mathbf{X}_{1j} \hat{\boldsymbol{\beta}}_j^t + \mathbf{p}_1 \hat{\rho}_j^t \right) \quad \text{by (24)} \quad (66)$$

$$= \sum_j b_{j\,new} \left(\mathbf{X}_{1j} \hat{\boldsymbol{\beta}}_{j\,new} + \mathbf{p}_1 \hat{\rho}_{j\,new} \right) + \sum_j \sum_{t=-1}^{-\infty} b_{j1}^t \left(\mathbf{X}_{1j} \hat{\boldsymbol{\beta}}_j^t + \mathbf{p}_1 \hat{\rho}_j^t \right) \qquad \text{split up} \qquad (67)$$

$$= w \sum_j b_j \left(\mathbf{X}_{1j} \hat{\boldsymbol{\beta}}_{j\,new} + \mathbf{p}_1 \hat{\rho}_{j\,new} \right) + (1-w) \sum_j \sum_{t=-1}^{-\infty} b_{j0}^t \left(\mathbf{X}_{1j} \hat{\boldsymbol{\beta}}_j^t + \mathbf{p}_1 \hat{\rho}_j^t \right) \quad \text{by (46),(41)} \quad (68)$$

$$= w \sum_j b_j \left(\mathbf{X}_{1j} \hat{\boldsymbol{\beta}}_{j\,new} + \mathbf{p}_1 \hat{\rho}_{j\,new} \right) + (1-w) \sum_j b_j \left(\mathbf{X}_{1j} \hat{\boldsymbol{\beta}}_{j0} + \mathbf{p}_1 \hat{\rho}_{j0} \right) \qquad \text{by (42),(43)} \quad (69)$$

Differentiate with respect to variables $w,\mathbf{p}$:

$$\begin{aligned} \mathbf{0} = {} & dw \sum_j b_j \mathbf{X}_{1j} \hat{\boldsymbol{\beta}}_{j\,new} - dw \sum_j b_j \mathbf{X}_{1j} \hat{\boldsymbol{\beta}}_{j0} \\ & + dw \sum_j b_j \mathbf{p}_1 \hat{\rho}_{j\,new} - dw \sum_j b_j \mathbf{p}_1 \hat{\rho}_{j0} \\ & + \mathbf{dp} \cdot w \sum_j b_j \hat{\rho}_{j\,new} + \mathbf{dp} \cdot (1-w) \sum_j b_j \hat{\rho}_{j0} \end{aligned} \qquad (70)$$

This can be evaluated at the point $w = 0$ (*new* investors are incremental); at this point

$$\mathbf{p}_1 = \mathbf{X}_1 \boldsymbol{\pi}_0 \qquad (71)$$

and
$$\sum_j b_j \mathbf{X}_{1j} \hat{\boldsymbol{\beta}}_{j0} + \sum_j b_j \mathbf{p}_1 \hat{\rho}_{j0} = \mathbf{0}_{T*1} \qquad \text{by (69)} \qquad (72)$$

Making these substitutions into (70) yields:

$$\mathbf{0} = dw \sum_j b_j \mathbf{X}_{1j} \hat{\boldsymbol{\beta}}_{j\,new} + dw \sum_j b_j \mathbf{p}_1 \hat{\rho}_{j\,new} + \mathbf{dp} \cdot \rho_0 \qquad \text{using (60)} \qquad (73)$$

$$= dw \sum_j b_j \begin{bmatrix} \mathbf{X}_1 \mathbf{a}_j & \mathbf{X}_1 \boldsymbol{\pi}_0 \end{bmatrix} \begin{bmatrix} \hat{\boldsymbol{\beta}}_{j\,new} \\ \hat{\rho}_{j\,new} \end{bmatrix} + \mathbf{dp} \cdot \rho_0 \qquad \text{by (71)} \qquad (74)$$

$$= dw \sum_j b_j \mathbf{X}_1 \tilde{\mathbf{a}}_{j0} \begin{bmatrix} \hat{\boldsymbol{\beta}}_{j\,new} \\ \hat{\rho}_{j\,new} \end{bmatrix} + \mathbf{dp} \cdot \rho_0 \qquad \text{by (29)} \qquad (75)$$

$$= dw \sum_j b_j \mathbf{X}_1 \tilde{\mathbf{a}}_{j0} \left(\tilde{\mathbf{a}}_{j0}' \tilde{\mathbf{a}}_{j0} \right)^{-1} \tilde{\mathbf{a}}_{j0}' \left(\boldsymbol{\mu} - \boldsymbol{\pi}_0 + \mathbf{e}_0 \right) + \mathbf{X}_1 \mathbf{d}\boldsymbol{\pi} \cdot \rho_0 \qquad \text{by (63)} \qquad (76)$$

$$\text{noting } \mathbf{dp} = \mathbf{X}_1 \boldsymbol{\pi}_1 - \mathbf{X}_0 \boldsymbol{\pi}_0 = \mathbf{X}_1 \mathbf{d}\boldsymbol{\pi} \text{ given (2)} \qquad (77)$$

Now this statement is true for all observations $\mathbf{x}_{1*N}$ which form the rows of dataset $\mathbf{X}$, and therefore is true of the coefficients alone:

$$\mathbf{0} = dw \sum_j b_j \tilde{\mathbf{a}}_{j0} \left(\tilde{\mathbf{a}}_{j0}' \tilde{\mathbf{a}}_{j0} \right)^{-1} \tilde{\mathbf{a}}_{j0}' \left(\boldsymbol{\mu} - \boldsymbol{\pi}_0 + \mathbf{e}_0 \right) + \mathbf{d}\boldsymbol{\pi} \cdot \rho_0 \qquad (78)$$

$$= dw \cdot \mathbf{H} \left(\boldsymbol{\mu} - \boldsymbol{\pi}_0 + \mathbf{e}_0 \right) + \mathbf{d}\boldsymbol{\pi} \cdot \rho_0 \qquad \# \qquad (79)$$

This theorem may appear to be a trivial technical result, but it is contended here that it underlies the efficiency of financial markets. Note what it does not say – the price change is given by the *change* in expectation, i.e.

$$\mathbf{d}\boldsymbol{\pi} = -\frac{dw}{\rho_0} \sum b_j \tilde{\mathbf{a}}_{j0} \left(\hat{\boldsymbol{\beta}}_{j\,new} - \hat{\boldsymbol{\beta}}_{j\,old} \right) \qquad (80)$$

but rather the theorem says:

$$\mathbf{d\pi} = -\frac{dw}{\rho_0}\sum b_j \tilde{\mathbf{a}}_{j0}\hat{\boldsymbol{\beta}}_{j\,new} \qquad \text{rearranging (78)} \qquad (81)$$

so the change in price is determined by the new estimates alone, and price is pushed in the direction of the underlying return parameter $\boldsymbol{\mu}$.

3.4. Properties of the estimation matrix

It is useful to introduce an orthogonalized and normalized version, $\boldsymbol{\alpha}_j$, of the augmented strategy matrix $\tilde{\mathbf{a}}_j$.

RESULT 9: NORMALIZED STRATEGY MATRIX. *The matrix* $\tilde{\mathbf{a}}_{j0}\left(\tilde{\mathbf{a}}_{j0}'\tilde{\mathbf{a}}_{j0}\right)^{-1}\tilde{\mathbf{a}}_{j0}'$ *can be expressed as* $\boldsymbol{\alpha}_{j0}\boldsymbol{\alpha}_{j0}'$ *where* $\boldsymbol{\alpha}_{j0}$ *is an* $N*jK^{aug}$ *matrix of orthogonalized and normalized strategy vectors.*

PROOF: The matrix $\tilde{\mathbf{a}}_{j0}\left(\tilde{\mathbf{a}}_{j0}'\tilde{\mathbf{a}}_{j0}\right)^{-1}\tilde{\mathbf{a}}_{j0}'$ is real symmetric so can be represented as $\boldsymbol{\Lambda}_a\boldsymbol{\lambda}_a\boldsymbol{\Lambda}_a'$ where $\boldsymbol{\lambda}_a$ is a diagonal matrix and $\boldsymbol{\Lambda}_a$ consists of real orthogonal eigenvectors. It is idempotent so all the eigenvalues must be 0 or 1. Multiplying by $\tilde{\mathbf{a}}_{j0}$ on either side of the matrix gives $\tilde{\mathbf{a}}_{j0}'\tilde{\mathbf{a}}_{j0}$ which has rank jK^{aug} so the matrix must have rank jK^{aug} not less. It therefore has jK^{aug} unity eigenvalues. Without loss of generality, place the unity eigenvalues and corresponding eigenvectors first in the matrix. So

$$\tilde{\mathbf{a}}_{j0}\left(\tilde{\mathbf{a}}_{j0}'\tilde{\mathbf{a}}_{j0}\right)^{-1}\tilde{\mathbf{a}}_{j0}' = \begin{bmatrix}\boldsymbol{\Lambda}_1 & \boldsymbol{\Lambda}_2\end{bmatrix}\begin{bmatrix}\mathbf{I}_{jKaug*jKaug} & \mathbf{0}\\ \mathbf{0} & \mathbf{0}\end{bmatrix}\begin{bmatrix}\boldsymbol{\Lambda}_1'\\ \boldsymbol{\Lambda}_2'\end{bmatrix} \qquad (82)$$

$$= \boldsymbol{\Lambda}_1\boldsymbol{\Lambda}_1' \qquad \text{Take } \boldsymbol{\alpha}_{j0} = \boldsymbol{\Lambda}_1. \qquad \# \qquad (83)$$

Observe that:

$$\boldsymbol{\alpha}_{j0}'\boldsymbol{\alpha}_{j0} = \boldsymbol{\Lambda}_1'\boldsymbol{\Lambda}_1 = \mathbf{I}_{jKaug*jKaug} \qquad (84)$$

as required. The estimation matrix **H** can be expressed as:

$$\mathbf{H}_0 = \sum_j b_j \boldsymbol{\alpha}_{j0}\boldsymbol{\alpha}_{j0}' \qquad (85)$$

It is also useful to partition $\boldsymbol{\alpha}_j$ into columns:

$$\boldsymbol{\alpha}_j = \begin{bmatrix}\mathbf{o}_{j1} & \mathbf{o}_{j2} & \end{bmatrix} \qquad (86)$$

RESULT 10. *The estimation matrix* $\mathbf{H}_0$ *is (i) can be expressed as* $\boldsymbol{\Lambda}_0\boldsymbol{\lambda}_0\boldsymbol{\Lambda}_0'$ *where* $\boldsymbol{\Lambda}_0$ *is an N*N matrix of orthogonal eigenvectors and* $\boldsymbol{\lambda}_0$ *is an N*N matrix of positive eigenvalues (ii) is positive definite.*

PROOF: By (9), $\{\mathbf{a}_{jk}\}$ spans the space $\mathbb{R}^N$. So for all non-zero N*1 vectors **x**, there exists an N*1 vector $\mathbf{a}_{Jk}$ such that:

$$\mathbf{x}'\mathbf{a}_{Jk} \neq 0 \qquad (87)$$

and corresponding normalized strategy matrix $\boldsymbol{\alpha}_{J0}$ which uses $\mathbf{a}_{Jk}$ such that:

$$\mathbf{x}_{N*1} = \boldsymbol{\alpha}_{J0}\boldsymbol{\beta}_{JK*1} + \boldsymbol{\varepsilon}_{N*1}, \tag{88}$$

where $\boldsymbol{\beta} \neq \mathbf{0}_{JK*1}$ (89)

and $\boldsymbol{\varepsilon}_{1*N}{}'\boldsymbol{\alpha}_{N*jKaug} = \mathbf{0}_{1*jKaug}$ (90)

Now
$$\mathbf{x}'\mathbf{H}_0\mathbf{x} = \mathbf{x}'\left(\sum_j b_j \boldsymbol{\alpha}_{j0}\boldsymbol{\alpha}_{j0}{}'\right)\mathbf{x} \tag{91}$$

$$\geq b_{J\,new}\mathbf{x}'\boldsymbol{\alpha}_{J0}\boldsymbol{\alpha}_{J0}{}'\mathbf{x} \qquad \boldsymbol{\alpha}_{j0}\boldsymbol{\alpha}_{j0}{}' \text{ is non-negative definite} \tag{92}$$

$$= b_{J\,new}\left(\boldsymbol{\beta}'\boldsymbol{\alpha}_{J0}{}' + \boldsymbol{\varepsilon}'\right)\boldsymbol{\alpha}_{J0}\boldsymbol{\alpha}_{J0}{}'\left(\boldsymbol{\alpha}_{J0}\boldsymbol{\beta} + \boldsymbol{\varepsilon}\right) \qquad \text{by (88)} \tag{93}$$

$$= b_{J\,new}\left(\boldsymbol{\alpha}_{J0}{}'\boldsymbol{\alpha}_{J0}\boldsymbol{\beta}\right)'_{1*JK}\left(\boldsymbol{\alpha}_{J0}{}'\boldsymbol{\alpha}_{J0}\boldsymbol{\beta}\right)_{JK*1} \qquad \text{by (90)} \tag{94}$$

$> \mathbf{0}$ as $\boldsymbol{\alpha}_{J0}{}'\boldsymbol{\alpha}_{J0}$ is full rank so multiplying by $\boldsymbol{\beta} \neq \mathbf{0}$ gives non-zero product. (95)

Further $\mathbf{H}_0$ is real symmetric so can be represented as $\boldsymbol{\Lambda}_0\boldsymbol{\lambda}_0\boldsymbol{\Lambda}_0{}'$ where $\boldsymbol{\lambda}_0$ is a diagonal matrix and $\boldsymbol{\Lambda}_0$ consists of real orthogonal eigenvectors. Since $\mathbf{H}_0$ is positive definite, $\boldsymbol{\lambda}_0$ must be positive diagonal. #

RESULT 11. *The eigenvalues $\boldsymbol{\lambda}_0$ of the estimation matrix $\mathbf{H}_0$ are such that*

$$0 < \boldsymbol{\lambda}_0 \leq 1 \tag{96}$$

PROOF: Let $\mathbf{v}$ be the eigenvector of $\mathbf{H}_0$ corresponding to eigenvalue λ.

$$\lambda^2\mathbf{v}'\mathbf{v} = \mathbf{v}'\mathbf{H}_0{}'\mathbf{H}_0\mathbf{v} \qquad \text{from definition of eigenvector} \tag{97}$$

$$= \sum_i\sum_j b_i b_j \hat{\mathbf{v}}_i{}'\hat{\mathbf{v}}_j \qquad \text{given } \mathbf{H}_0\mathbf{v} = \sum_j b_j\boldsymbol{\alpha}_{j0}\boldsymbol{\alpha}_{j0}{}'\mathbf{v} \tag{98}$$

$$= \sum_j b_j\hat{\mathbf{v}}_j \tag{99}$$

$$\leq \sum_i\sum_j b_i b_j \mathbf{v}'\mathbf{v} \qquad \text{given } \hat{\mathbf{v}}_i{}'\hat{\mathbf{v}}_j = \left|\hat{\mathbf{v}}_i\right|\cdot\left|\hat{\mathbf{v}}_j\right|\cdot\cos\theta \tag{100}$$

$$\leq |\mathbf{v}|\cdot|\mathbf{v}| \quad \text{since } \left|\hat{\mathbf{v}}_j\right| \leq |\mathbf{v}| \tag{101}$$

$$= \mathbf{v}'\mathbf{v} \qquad \text{noting } \sum_i\sum_j b_i b_j = 1 \tag{102}$$

Hence $\lambda^2 \leq 1$ dividing both sides by $\mathbf{v}'\mathbf{v} = 1$ # (103)

3.5. The Efficient Market Theorem

The estimation matrix $\mathbf{H}_0$ is a function of the augmented strategy matrix $\tilde{\mathbf{a}}_{j0}$ which is a function of the price vector $\boldsymbol{\pi}_0$. Since price varies, it is useful to develop an approximation for $\mathbf{H}_0(\boldsymbol{\mu} - \boldsymbol{\pi}_0)$.

LEMMA 12. *A first order approximation of $\mathbf{H}_0(\boldsymbol{\mu} - \boldsymbol{\pi}_0)$ is $\mathbf{H}_\mu(\boldsymbol{\mu} - \boldsymbol{\pi}_0)$, where $\mathbf{H}_\mu$ denotes $\mathbf{H}_0$ evaluated at $\boldsymbol{\pi}_0 = \boldsymbol{\mu}$.*

PROOF: $\mathbf{H} = \sum_{j\,new} b_j \boldsymbol{\alpha}_j \boldsymbol{\alpha}_j{}'$ by (85) (104)

$$= \sum_j b_j \begin{bmatrix} \mathbf{o}_{j1} & \mathbf{o}_{j2} & \end{bmatrix} \begin{bmatrix} \mathbf{o}_{j1}{}' \\ \mathbf{o}_{j2}{}' \\ \end{bmatrix} \quad \text{by (86)} \tag{105}$$

$$= \sum_j b_j \sum_k \mathbf{o}_{jk} \mathbf{o}_{jk}{}' \tag{106}$$

Differentiate $\mathbf{H(\boldsymbol{\mu} - \boldsymbol{\pi})}$ with respect to $\boldsymbol{\pi}$ (in the following, $\mathbf{c}_1$ and $\mathbf{c}_2$ are constants):

$$\frac{\partial \mathbf{H}(\boldsymbol{\mu} - \boldsymbol{\pi})}{\partial \boldsymbol{\pi}} = \sum_j b_j \sum_k \frac{\partial}{\partial \boldsymbol{\pi}} \left(\mathbf{o}_{jk} \mathbf{o}_{jk}{}' (\boldsymbol{\mu} - \boldsymbol{\pi}) \right) \tag{107}$$

$$\frac{\partial}{\partial \boldsymbol{\pi}} \left(\mathbf{o}_{jk} \mathbf{o}_{jk}{}' (\boldsymbol{\mu} - \boldsymbol{\pi}) \right) = \frac{\partial \mathbf{o}_{jk}}{\partial \boldsymbol{\pi}} \left(\mathbf{o}_{jk}{}' (\boldsymbol{\mu} - \boldsymbol{\pi}) \right) + \frac{\partial}{\partial \boldsymbol{\pi}} \left(\mathbf{c}_1 \cdot \mathbf{o}_{jk}{}' \cdot \mathbf{c}_2 \right) \Big|_{\substack{\mathbf{c}_1 = \mathbf{o}_{jk} \\ \mathbf{c}_2 = \boldsymbol{\mu} - \boldsymbol{\pi}}} + \mathbf{o}_{jk} \mathbf{o}_{jk}{}' \frac{\partial}{\partial \boldsymbol{\pi}} (\boldsymbol{\mu} - \boldsymbol{\pi}) \tag{108}$$

$$= \frac{\partial \mathbf{o}_{jk}}{\partial \boldsymbol{\pi}} \left(\mathbf{o}_{jk}{}' (\boldsymbol{\mu} - \boldsymbol{\pi}) \right) + \frac{\partial}{\partial \boldsymbol{\pi}} \left(\mathbf{c}_1 \cdot \mathbf{c}_2{}' \cdot \mathbf{o}_{jk} \right) \Big|_{\substack{\mathbf{c}_1 = \mathbf{o}_{jk} \\ \mathbf{c}_2 = \boldsymbol{\mu} - \boldsymbol{\pi}}} + \mathbf{o}_{jk} \mathbf{o}_{jk}{}' (\mathbf{-I}) \quad \text{interchanging } \mathbf{c}_2, \mathbf{o}_{jk} \tag{109}$$

$$= \frac{\partial \mathbf{o}_{jk}}{\partial \boldsymbol{\pi}} \left(\mathbf{o}_{jk}{}' (\boldsymbol{\mu} - \boldsymbol{\pi}) \right) + \mathbf{o}_{jk} (\boldsymbol{\mu} - \boldsymbol{\pi})' \frac{\partial \mathbf{o}_{jk}}{\partial \boldsymbol{\pi}} - \mathbf{o}_{jk} \mathbf{o}_{jk}{}' \tag{110}$$

so
$$\frac{\partial \mathbf{H}(\boldsymbol{\mu} - \boldsymbol{\pi})}{\partial \boldsymbol{\pi}} = \sum_j b_j \sum_k \left(\frac{\partial \mathbf{o}_{jk}}{\partial \boldsymbol{\pi}} \left(\mathbf{o}_{jk}{}' (\boldsymbol{\mu} - \boldsymbol{\pi}) \right) + \mathbf{o}_{jk} (\boldsymbol{\mu} - \boldsymbol{\pi})' \frac{\partial \mathbf{o}_{jk}}{\partial \boldsymbol{\pi}} - \mathbf{o}_{jk} \mathbf{o}_{jk}{}' \right) \tag{111}$$

and the required first order approximation is given by

$$\mathbf{H}(\boldsymbol{\mu} - \boldsymbol{\pi}) \approx \mathbf{H}(\boldsymbol{\mu} - \boldsymbol{\pi})\big|_{\boldsymbol{\pi} = \boldsymbol{\mu}} + \frac{\partial \mathbf{H}(\boldsymbol{\mu} - \boldsymbol{\pi})}{\partial \boldsymbol{\pi}} \bigg|_{\boldsymbol{\pi} = \boldsymbol{\mu}} (\boldsymbol{\pi} - \boldsymbol{\mu}) \tag{112}$$

noting the final factor is $\boldsymbol{\pi} - \boldsymbol{\mu}$, as per a Taylor series, not $\boldsymbol{\mu} - \boldsymbol{\pi}$

$$= \mathbf{0} + \sum_j b_j \sum_k \left(\mathbf{0} + \mathbf{0} - \mathbf{o}_{jk} \mathbf{o}_{jk}{}' \right) \Big|_{\boldsymbol{\pi} = \boldsymbol{\mu}} (\boldsymbol{\pi} - \boldsymbol{\mu}) \quad \text{using (111)} \tag{113}$$

$$= -\mathbf{H}_\mu (\boldsymbol{\pi} - \boldsymbol{\mu}) \tag{114}$$

$$= \mathbf{H}_\mu (\boldsymbol{\mu} - \boldsymbol{\pi}) \quad \# \tag{115}$$

This technical result is straightforward. A first order approximation is quite accurate in the region around the mean $\boldsymbol{\mu}$. Consideration of the underlying process suggests that points in the other, outlying regions will be rapidly pushed into the region where the approximation is good.

Let $\bar{\bar{\rho}}$ denote the harmonic mean of price coefficient ρ.

$$\bar{\bar{\rho}} = \left[E \left[\frac{1}{\rho} \right] \right]^{-1} \quad \text{ie} \quad \frac{1}{\bar{\bar{\rho}}} = E \left[\frac{1}{\rho} \right] \tag{116}$$

LEMMA 13. *The eigenvalues* $\dot{\boldsymbol{\lambda}}$ *and eigenvectors* $\dot{\mathbf{v}}$ *of the matrix* $\mathbf{I} + \frac{dw}{\bar{\bar{\rho}}} \mathbf{H}_\mu$ *are given by*

$$\dot{\boldsymbol{\lambda}} = 1 + \frac{dw}{\bar{\bar{\rho}}}\boldsymbol{\lambda} \tag{117}$$

$$\dot{\boldsymbol{\Lambda}} = \boldsymbol{\Lambda} \tag{118}$$

where $\boldsymbol{\lambda}$, $\boldsymbol{\Lambda}$ *denote the eigenvalues and eigenvectors of the estimation matrix* $\mathbf{H}_\mu$.

PROOF: $\left[\mathbf{I} + \frac{dw}{\bar{\bar{\rho}}}\mathbf{H}_\mu\right] \cdot \boldsymbol{\Lambda} = \boldsymbol{\Lambda}\left[\mathbf{I} + \frac{dw}{\bar{\bar{\rho}}}\boldsymbol{\lambda}\right]$ # (119)

LEMMA 14. *If stability condition* (125) *is satisfied, any eigenvalues* $\dot{\lambda}$ *of the matrix* $\mathbf{I} + \frac{dw}{\bar{\bar{\rho}}} \cdot \mathbf{H}_\mu$ *falls in the range:*

$$0 < \dot{\lambda} < 1 \tag{120}$$

PROOF: $1 > -\frac{dw \cdot \lambda}{\bar{\bar{\rho}}}$ rearranging (125), true for all λ, and noting $\bar{\bar{\rho}} < 0$ (121)

so $0 < -\frac{dw \cdot \lambda}{\bar{\bar{\rho}}} < 1$ expression is positive (122)

$1 > 1 + \frac{dw \cdot \lambda}{\bar{\bar{\rho}}} > 0$ multiply by -1, add 1 (123)

$0 < \dot{\lambda} < 1$ by (117) # (124)

Two features of the model are required here but not given a full treatment until the following section:

- *Expectational stability condition:*

  $$\rho < -dw \cdot \lambda^{\max} \tag{125}$$

  This condition is expressed in terms of the endogenous variable $\lambda^{\max}$, the maximum eigenvalue of $\mathbf{H}_\mu$. A statement of the condition in terms of exogenous variables is developed in section 4 below. The reader may consider that to require a stability premise for the results below is to beg the question – market stability is a property which should be proven within the framework of the model. This view presupposes that real world markets are stable and this is not always the case. One way of explaining market bubbles is by violation of the stability condition.
- *Practical independence of the price coefficient:* ρ_0 is independent of current price $\boldsymbol{\pi}_0$. This property cannot be literally true as the price coefficient and price are endogenous variables and there is only one source of noise in the model. Nonetheless it is shown in section 4 that it is approximately true both theoretically and in simulation.

It is now possible to prove the central result of this paper.

THEOREM 15: THE EFFICIENT MARKET THEOREM. *If the stability condition* (125) *is satisfied then:*

$$\lim_{n\to\infty} E\left(\boldsymbol{\pi}_n\right) = \boldsymbol{\mu} \tag{126}$$

PROOF: $\mathbf{d\pi} = -\frac{dw}{\rho_0}\mathbf{H}_0\left(\boldsymbol{\mu} - \boldsymbol{\pi}_0 + \mathbf{e}_0\right)$ Equation (62) (127)

$$\boldsymbol{\mu} - \boldsymbol{\pi}_1 = \boldsymbol{\mu} - \boldsymbol{\pi}_0 + \frac{dw}{\rho_0}\mathbf{H}_0\left(\boldsymbol{\mu} - \boldsymbol{\pi}_0 + \mathbf{e}_0\right) \quad \text{rearranging} \tag{128}$$

$$\approx \boldsymbol{\mu} - \boldsymbol{\pi}_0 + \frac{dw}{\rho_0}\mathbf{H}_\mu\left(\boldsymbol{\mu} - \boldsymbol{\pi}_0 + \mathbf{e}_0\right) \quad \text{applying Lemma 8} \tag{129}$$

Take expectations given ρ_0 is taken as independent of $\boldsymbol{\pi}$ and $\mathbf{H}_\mu$ is constant:

$$E\left[\boldsymbol{\mu} - \boldsymbol{\pi}_1\right] = E\left[\boldsymbol{\mu} - \boldsymbol{\pi}_0\right] + dw \cdot E\left[\frac{1}{\rho_0}\right]\mathbf{H}_\mu\, E\left[\boldsymbol{\mu} - \boldsymbol{\pi}_0\right] \quad \text{noting } E\left[\mathbf{e}_0\right] = \mathbf{0} \tag{130}$$

$$= \left(\mathbf{I} + \frac{dw}{\bar{\bar{\rho}}}\mathbf{H}_\mu\right)E\left[\boldsymbol{\mu} - \boldsymbol{\pi}_0\right] \quad \text{(117), factorizing} \tag{131}$$

$$= \boldsymbol{\Lambda}\dot{\boldsymbol{\lambda}}\boldsymbol{\Lambda}' E\left[\boldsymbol{\mu} - \boldsymbol{\pi}_0\right] \quad \text{substituting } \boldsymbol{\Lambda}\dot{\boldsymbol{\lambda}}\boldsymbol{\Lambda}' \text{ for } \mathbf{I} + \frac{dw}{\bar{\bar{\rho}}}\mathbf{H}_\mu \text{ by Lemma 9} \tag{132}$$

so $E\left[\boldsymbol{\mu} - \boldsymbol{\pi}_2\right] = \boldsymbol{\Lambda}\dot{\boldsymbol{\lambda}}^2\boldsymbol{\Lambda}' E\left[\boldsymbol{\mu} - \boldsymbol{\pi}_0\right]$ orthogonal eigenvectors (133)

and result follows by Lemma 10. #

For a stock which has traded for a few months after its initial listing, this result can be restated more simply as

$$E\left[\boldsymbol{\pi}\right] = \boldsymbol{\mu} \tag{134}$$

The process of convergence is illustrated in the following diagram:

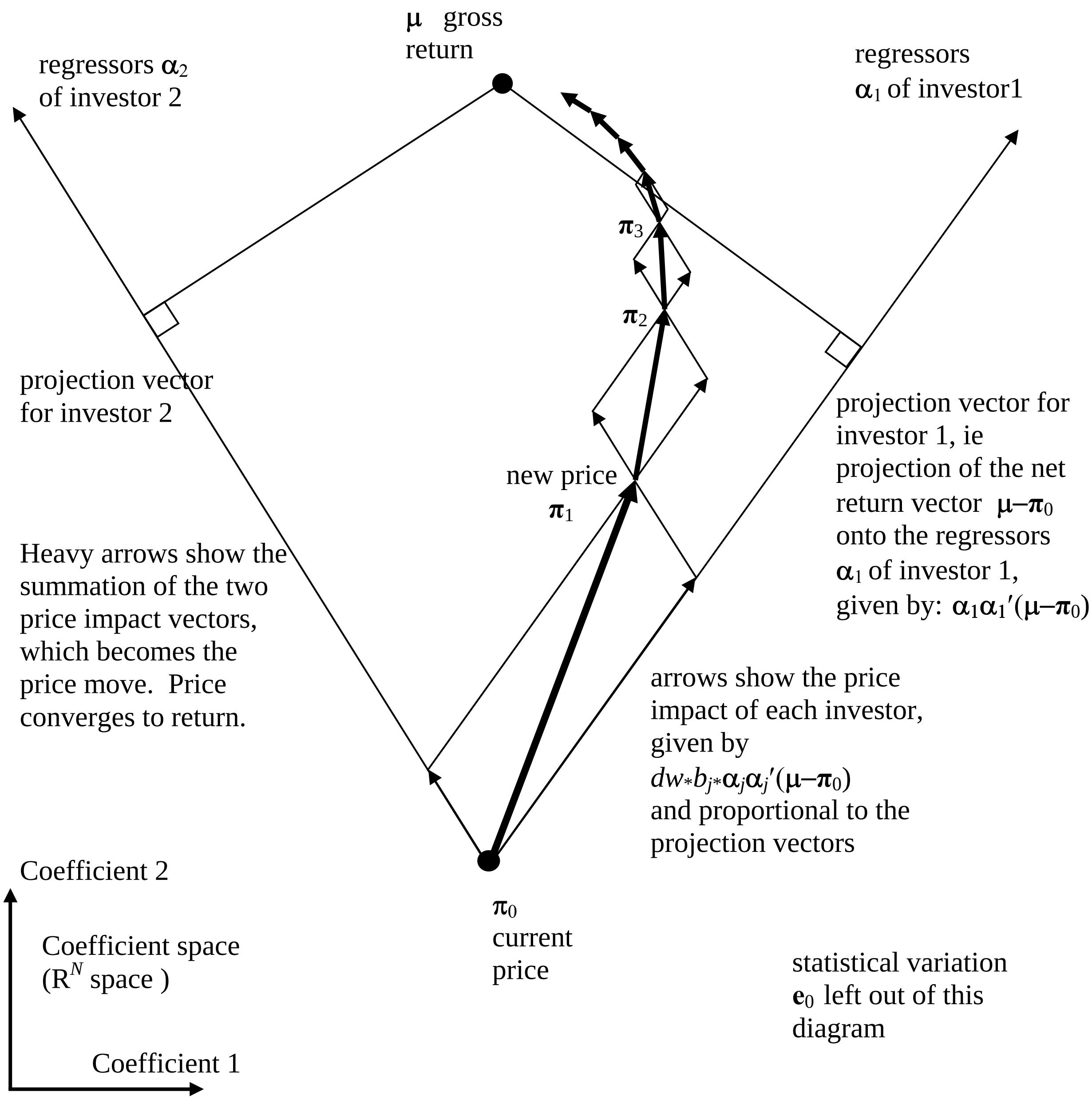


FIG. 1. Convergence of price to return in the objective model. Investor j regresses the net return $\boldsymbol{\mu} - \boldsymbol{\pi}$ against a private set of regressors $\boldsymbol{\alpha}_j$. This corresponds to the projection of the net return vector onto the regressor vector. The investor then buys according to this information, which moves the price along the line of the regressor – this is shown by the unbolded arrows. The total impact on price is the vector summation of the individual impacts of each investor and this is shown by the bold arrows. Successive iterations of this regression process move price along the path of the bold arrows to converge with return.

3.6. Consistency of the price estimator

Consider the matrix $-\frac{dw}{\rho_0}\mathbf{H}_\mu$. This matrix has eigenvector matrix $\mathbf{\Lambda}$ and eigenvalue matrix $\ddot{\boldsymbol{\lambda}}_0$ given by

$$\ddot{\boldsymbol{\lambda}}_0 = -\frac{dw}{\rho_0}\boldsymbol{\lambda} \qquad \text{proof as per Lemma 3.8} \qquad (135)$$

$\ddot{\boldsymbol{\lambda}}$ without a time subscript denotes the eigenvalue matrix of $-\frac{dw}{\overline{\overline{\rho}}}\mathbf{H}_\mu$.

Observe $E\left[\ddot{\boldsymbol{\lambda}}_0\right] = E\left[-\frac{dw}{\rho_0}\boldsymbol{\lambda}\right] = -\frac{dw}{\overline{\overline{\rho}}}\boldsymbol{\lambda} = \ddot{\boldsymbol{\lambda}}$ (136)

and $\dot{\boldsymbol{\lambda}}_0 + \ddot{\boldsymbol{\lambda}}_0 = \left[\mathbf{I} + \frac{dw}{\rho_0}\boldsymbol{\lambda}\right] - \frac{dw}{\rho_0}\boldsymbol{\lambda} = \mathbf{I}$ by (117) and (135) (137)

$\ddot{\boldsymbol{\lambda}}_0$ is the same order of magnitude as $\boldsymbol{\lambda}$, small, unlike $\dot{\boldsymbol{\lambda}}_0 = \mathbf{I} - \ddot{\boldsymbol{\lambda}}_0$ which is roughly unity. Now define the variance of price $\mathbf{\Sigma}_{N*N}$ by

$$\mathbf{\Sigma}_{N*N} = E\left[\overline{\boldsymbol{\mu}-\boldsymbol{\pi}}\;\overline{\boldsymbol{\mu}-\boldsymbol{\pi}}'\right] \qquad (138)$$

where the vincula indicate coordinates of the variable relative to the orthogonal basis defined by the eigenvectors, derived by multiplying by $\mathbf{\Lambda}'$. The variance of the error term transformed to the eigenvector coordinates, $\overline{\mathbf{e}}$, is still σ^2:

$$E\left[\overline{\mathbf{e}}_0\,\overline{\mathbf{e}}_0'\right] = E\left[\mathbf{\Lambda}'\mathbf{e}_0\mathbf{e}_0'\mathbf{\Lambda}\right] = \mathbf{\Lambda}'\sigma^2\mathbf{I}\mathbf{\Lambda} = \sigma^2\mathbf{I}_{N*N}\,. \qquad (139)$$

System equilibrium requires that the expected variance of price around parameter is constant with respect to time, i.e.

$$\mathbf{\Sigma}_0 = \mathbf{\Sigma}_1 \qquad (140)$$

RESULT 16. *If $\dot{\boldsymbol{\lambda}}_t, \ddot{\boldsymbol{\lambda}}_t$ are taken as constants equal to the mean values, at equilibrium the variance of price is given by:*

$$\mathbf{\Sigma} = \sigma^2\left(\mathbf{I}-\dot{\boldsymbol{\lambda}}\right)\left(\mathbf{1}+\dot{\boldsymbol{\lambda}}\right)^{-1} = \sigma^2\cdot\ddot{\boldsymbol{\lambda}}\left(\mathbf{2I}-\ddot{\boldsymbol{\lambda}}\right)^{-1} \qquad (141)$$

PROOF: $\boldsymbol{\mu}-\boldsymbol{\pi}_1 = \left(\mathbf{I}+\frac{dw}{\rho_0}\mathbf{H}_\mu\right)\left(\boldsymbol{\mu}-\boldsymbol{\pi}_0\right) + \frac{dw}{\rho_0}\mathbf{H}_\mu\mathbf{e}_0$ by (129) (142)

$$\boldsymbol{\mu}-\boldsymbol{\pi}_1 = \mathbf{\Lambda}\dot{\boldsymbol{\lambda}}_0\mathbf{\Lambda}'\left(\boldsymbol{\mu}-\boldsymbol{\pi}_0\right) - \mathbf{\Lambda}\ddot{\boldsymbol{\lambda}}_0\mathbf{\Lambda}'\mathbf{e}_0 \qquad (143)$$

$$\overline{\boldsymbol{\mu}-\boldsymbol{\pi}_1} = \left(\mathbf{I}-\ddot{\boldsymbol{\lambda}}_0\right)\overline{\boldsymbol{\mu}-\boldsymbol{\pi}_0} - \ddot{\boldsymbol{\lambda}}_0\overline{\mathbf{e}_0} \qquad (144)$$

Multiplying each side by itself:

$$\overline{\boldsymbol{\mu}-\boldsymbol{\pi}_1}\;\overline{\boldsymbol{\mu}-\boldsymbol{\pi}_1}' = \left(\mathbf{I}-\ddot{\boldsymbol{\lambda}}_0\right)\overline{\boldsymbol{\mu}-\boldsymbol{\pi}_0}\;\overline{\boldsymbol{\mu}-\boldsymbol{\pi}_0}'\left(\mathbf{I}-\ddot{\boldsymbol{\lambda}}_0\right) - 2\left(\mathbf{I}-\ddot{\boldsymbol{\lambda}}_0\right)\overline{\boldsymbol{\mu}-\boldsymbol{\pi}_0}\;\overline{\mathbf{e}_0}'\ddot{\boldsymbol{\lambda}}_0 + \ddot{\boldsymbol{\lambda}}_0\overline{\mathbf{e}_0}\;\overline{\mathbf{e}_0}'\ddot{\boldsymbol{\lambda}}_0 \qquad (145)$$

$$E\left[\overline{\boldsymbol{\mu}-\boldsymbol{\pi}_1}\;\overline{\boldsymbol{\mu}-\boldsymbol{\pi}_1}'\right] = \left(\mathbf{I}-\ddot{\boldsymbol{\lambda}}\right)E\left[\overline{\boldsymbol{\mu}-\boldsymbol{\pi}_0}\;\overline{\boldsymbol{\mu}-\boldsymbol{\pi}_0}'\right]\left(\mathbf{I}-\ddot{\boldsymbol{\lambda}}\right) + \ddot{\boldsymbol{\lambda}}\,E\left[\overline{\mathbf{e}_0}\;\overline{\mathbf{e}_0}'\right]\ddot{\boldsymbol{\lambda}} \qquad (146)$$

noting $\overline{\boldsymbol{\mu}-\boldsymbol{\pi}_0}$, $\overline{\mathbf{e}_0}$ are independent so the expectation of the second term is 0.

$$\mathbf{\Sigma} - \left(\mathbf{I}-\ddot{\boldsymbol{\lambda}}\right)\mathbf{\Sigma}\left(\mathbf{I}-\ddot{\boldsymbol{\lambda}}\right) = \sigma^2\ddot{\boldsymbol{\lambda}}^2 \qquad \text{by (138), (139), (140)} \qquad (147)$$

Consider element Σ_{ij} of these $N*N$ matrices, $i \neq j$:

$$\Sigma_{ij} - \dot{\lambda}_i \Sigma_{ij} \dot{\lambda}_j = 0, \quad i \neq j \qquad \text{recalling that } \dot{\lambda} + \ddot{\lambda} = 1 \qquad (148)$$

$$\left(1 - \dot{\lambda}_i \dot{\lambda}_j\right) \Sigma_{ij} = 0 \qquad (149)$$

$$\Sigma_{ij} = 0 \qquad \text{given } 0 < \dot{\lambda}_i, \dot{\lambda}_j < 1 \text{ by (120)} \qquad (150)$$

So $\mathbf{\Sigma}$ is a diagonal matrix. Rewriting (147) given diagonal matrices are commutative:

$$\mathbf{\Sigma}\left(\mathbf{I} - \left(\mathbf{I} - \ddot{\boldsymbol{\lambda}}\right)\left(\mathbf{I} - \ddot{\boldsymbol{\lambda}}\right)\right) = \sigma^2 \ddot{\boldsymbol{\lambda}}^2 \qquad (151)$$

$$\mathbf{\Sigma} = \sigma^2 \ddot{\boldsymbol{\lambda}}^2 \left(\mathbf{I} - \left(\mathbf{I} - \ddot{\boldsymbol{\lambda}}\right)\left(\mathbf{I} - \ddot{\boldsymbol{\lambda}}\right)\right)^{-1} = \sigma^2 \ddot{\boldsymbol{\lambda}}\left(2\mathbf{I} - \ddot{\boldsymbol{\lambda}}\right)^{-1} = \sigma^2 \left(\mathbf{I} - \dot{\boldsymbol{\lambda}}\right)\left(\mathbf{1} + \dot{\boldsymbol{\lambda}}\right)^{-1} \ \# \qquad (152)$$

This result establishes the consistency of price $\boldsymbol{\pi}$ as an estimator of return, although this is disguised. Normally the variance of an OLS estimator is given by $\sigma^2 \left(\mathbf{X}'\mathbf{X}\right)^{-1}$ and the factor $\left(\mathbf{X}'\mathbf{X}\right)^{-1}$ increases as $T \to \infty$. Here $\left(\mathbf{X}'\mathbf{X}\right)^{-1}$ is fixed at $\mathbf{I}_{N*N}$ and the variance $\mathbf{\Sigma}$ is also fixed as T increases. As $T \to \infty$ the data is scaled downwards to maintain the identity condition, and the coefficients $\boldsymbol{\mu}, \boldsymbol{\pi}$ are correspondingly scaled upwards. The ratio of variance to variable decreases to zero as $T \to \infty$ as expected.

In fact $\dot{\boldsymbol{\lambda}}_t, \ddot{\boldsymbol{\lambda}}_t$ are not constant, but simulation shows that the variance is small relative to the mean so the above result is a reasonable approximation.

3.7. Simulation

As results depend on first order approximations the objective model was simulation tested. Return is assumed to be a function of two parameters, and there are two classes of investor. The first type look at data only and the second type look at price only.

		Investor 1: data only	Investor 2: price only
Relative proportion	b_j	0.5	0.5
Augmented strategy matrix	$\tilde{\mathbf{a}}_j$	$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$	$\boldsymbol{\pi}_{2*1}$
Regression coefficient using (34)	$\hat{\boldsymbol{\beta}}_j$	$\hat{\boldsymbol{\beta}} = \boldsymbol{\mu}_{2*1} - \boldsymbol{\pi}_{2*1} + \mathbf{e}_{2*1}$	$\hat{\rho} = \left(\boldsymbol{\pi}'\boldsymbol{\pi}\right)^{-1} \boldsymbol{\pi}'(\boldsymbol{\mu}_{2*1} - \boldsymbol{\pi}_{2*1} + \mathbf{e}_{2*1})$

TABLE 1: Characterization of investor types used in the simulation.

Estimates $\hat{\boldsymbol{\beta}}_1, \rho_1$ are updated as per (52),(59) and price is calculated by $\boldsymbol{\pi}_1 = -\frac{\hat{\boldsymbol{\beta}}_1}{\rho_1}$. There are no approximations in this stratagem. Typical results are shown in Figure 2. The price coefficient converges to a negative value providing the update proportion dw is kept low. This is necessary because of the low degrees of freedom $N = 2$ and is explained in Section 6 (high r_{error}). Parameters of the simulation shown below are:

$$\boldsymbol{\mu} = \begin{bmatrix} 0.2 \\ 0.3 \end{bmatrix};\ \boldsymbol{\sigma}_e = \begin{bmatrix} 0.10 \\ 0.05 \end{bmatrix};\ \hat{\boldsymbol{\beta}}_{x0} = \begin{bmatrix} 0.000012 \\ 0.000001 \end{bmatrix};\ \hat{\rho}_0 = -0.00002;\ \boldsymbol{\pi}_0 = \begin{bmatrix} 0.6 \\ 0.1 \end{bmatrix};\ dw = 0.000001$$

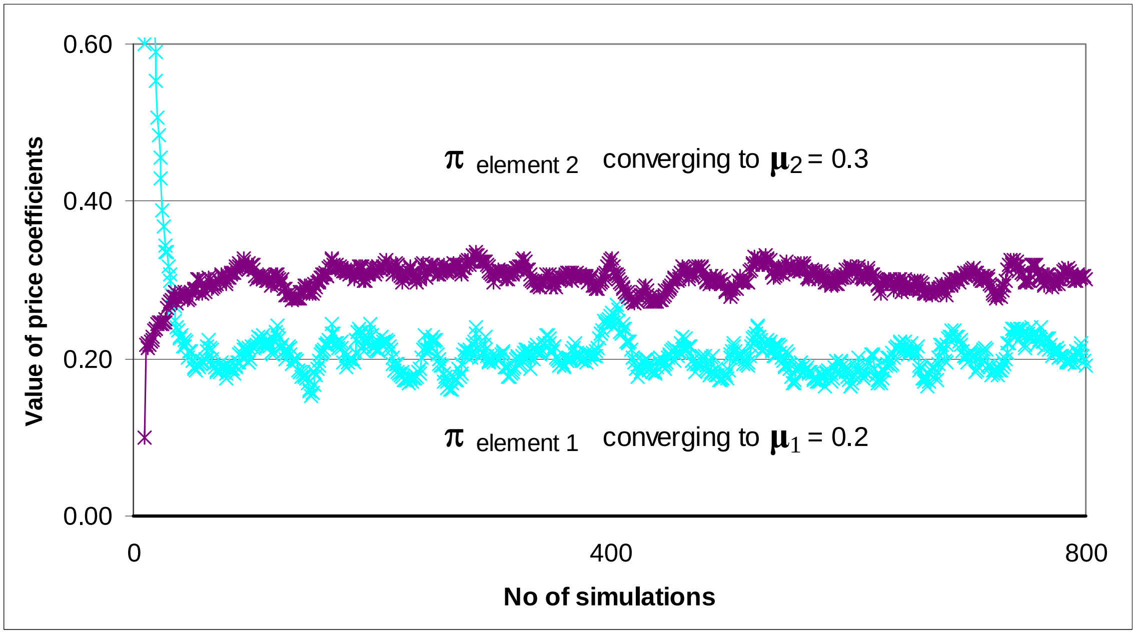


FIG. 2. Simulation showing convergence of price vector $\boldsymbol{\pi}$ to return vector $\boldsymbol{\mu}$.

4. The price coefficient

4.1. Price change theorem under an even distribution assumption

The most basic requirement of a market model is that it is capable of making a price, and to that end we need to demonstrate that the price coefficient $\hat{\rho}$ used in the price equation (22) will be negative. Prima facie it appears that regression of net return $\mathbf{y} - \mathbf{p} \approx \mathbf{X}\boldsymbol{\mu} - \mathbf{X}\boldsymbol{\pi}$ against price $\mathbf{X}\boldsymbol{\pi}$ will eventually yield a result close to zero, and this is borne out by Result 3.4.1. Price has so far been treated the same as any other variable, but it has special properties which must be studied in order to understand how it can be negative.

Note that the situation where price is negative (insurance policies) requires a negative price coefficient in the same way as the ordinary case. At zero price all investors will want to go short as a negative payment is expected. As price goes negative, a negative price coefficient is needed to generate a positive expected return. Demand and supply are not mirror images of normal; they are the same as normal but displaced downwards.

We can divide the estimation matrix between those investors who include price in the regression, and those who don't.

$$\mathbf{H} = \sum_{\substack{j \\ exclude}} b_j \mathbf{a}_j \left(\mathbf{a}_j{}'\mathbf{a}_j\right)^{-1} \mathbf{a}_j{}' + \sum_{\substack{j \\ include}} b_j \mathbf{a}_j \left(\mathbf{a}_j{}'\mathbf{a}_j\right)^{-1} \mathbf{a}_j{}' \tag{153}$$

$$= \mathbf{H}_{exclude} + \mathbf{H}_{include} \tag{154}$$

PREMISE 9: EVEN DISTRIBUTION. We assume that the non-price regressors are evenly distributed so that the eigenvalues λ_X are equal.

$$\mathbf{H}_{exclude} = \boldsymbol{\Lambda}_{exclude} \begin{bmatrix} \lambda_{\substack{X \\ exclude}} & & \\ & \lambda_{\substack{X \\ exclude}} & \\ & & \lambda_{\substack{X \\ exclude}} \end{bmatrix} \boldsymbol{\Lambda}_{exclude}' = \lambda_{\substack{X \\ exclude}} \mathbf{I}_{N*N} \tag{155}$$

Where price is included in a regression it is an eigenvector of the projection matrix and its eigenvalue is unity. Price must be an eigenvector of $\mathbf{H}_{include}$, which is the sum of projection matrices containing price, and its eigenvalue is the sum of the proportions:

$$\lambda_{\substack{P \\ include}} = \sum_{\substack{j \\ include}} b_j = b_{price} \tag{156}$$

Pursuant to the even distribution assumption the non-price eigenvalues are taken as equal. A non-price eigenvector of $\mathbf{H}_{include}$ cannot have eigenvalue greater than one for any one projection matrix, so its eigenvalue cannot be greater than b_{price}.

$$\mathbf{H}_{include} = \left[\boldsymbol{\pi}\left(\boldsymbol{\pi}'\boldsymbol{\pi}\right)^{-0.5} \quad \mathbf{v}_2 \quad \mathbf{v}_3 \right] \begin{bmatrix} \lambda_{\substack{P \\ include}} & & \\ & \lambda_{\substack{X \\ include}} & \\ & & \lambda_{\substack{X \\ include}} \end{bmatrix} \begin{bmatrix} \left(\boldsymbol{\pi}'\boldsymbol{\pi}\right)^{-0.5} \boldsymbol{\pi}' \\ \mathbf{v}_2 \\ \mathbf{v}_3 \end{bmatrix} \tag{157}$$

$$= \lambda_{\substack{X \\ include}} \mathbf{I}_{N*N} + \left(\lambda_{\substack{P \\ include}} - \lambda_{\substack{X \\ include}} \right) \boldsymbol{\pi}\left(\boldsymbol{\pi}'\boldsymbol{\pi}\right)^{-1} \boldsymbol{\pi}' \tag{158}$$

where $\lambda_{\substack{P \\ include}} - \lambda_{\substack{X \\ include}} \geq 0$ (159)

So in total:

$$\mathbf{H} = \lambda_{\substack{X \\ exclude}} \mathbf{I}_{N*N} + \lambda_{\substack{X \\ include}} \mathbf{I}_{N*N} + \left(\lambda_{\substack{P \\ include}} - \lambda_{\substack{X \\ include}} \right) \boldsymbol{\pi}\left(\boldsymbol{\pi}'\boldsymbol{\pi}\right)^{-1} \boldsymbol{\pi}' \tag{160}$$

$$= \lambda_X \mathbf{I}_{N*N} + \left(\lambda_P - \lambda_X \right) \boldsymbol{\pi}\left(\boldsymbol{\pi}'\boldsymbol{\pi}\right)^{-1} \boldsymbol{\pi}' \tag{161}$$

where $\lambda_X = \lambda_{\substack{X \\ exclude}} + \lambda_{\substack{X \\ include}}$ (162)

$\lambda_P = \lambda_{\substack{X \\ exclude}} + \lambda_{\substack{P \\ include}}$ note $\lambda_P \geq b_{price}$ (163)

and $\lambda_P - \lambda_X \geq 0$ (164)

RESULT 27: PRICE CHANGE THEOREM- EVEN DISTRIBUTION VERSION.

$$\mathbf{d\boldsymbol{\pi}} = -\frac{dw}{\rho_0} \left(\lambda_X \left(\boldsymbol{\mu} - \boldsymbol{\pi}_0 + \mathbf{e}_0 \right) + \left(\lambda_P - \lambda_X \right) \hat{\rho}_{\substack{new \\ simple}} \boldsymbol{\pi} \right) \tag{165}$$

where $\hat{\rho}_{\substack{new \\ simple}} = \left(\boldsymbol{\pi}'\boldsymbol{\pi}\right)^{-1} \boldsymbol{\pi}' \left(\boldsymbol{\mu} - \boldsymbol{\pi} + \mathbf{e}_0 \right)$ (166)

denotes the result of simple regression of return on price. Simple regression means that return is regressed on price alone, as distinct from multiple regression where other variables are included. A constant term is not included in the simple regression.

PROOF: $\mathbf{d\pi} = -\dfrac{dw}{\rho_0}\Big(\lambda_X \mathbf{I}_{N*N} + \left(\lambda_P - \lambda_X\right)\boldsymbol{\pi}\left(\boldsymbol{\pi}'\boldsymbol{\pi}\right)^{-1}\boldsymbol{\pi}'\Big)\left(\boldsymbol{\mu} - \boldsymbol{\pi} + \mathbf{e}_0\right)$ apply (161) to (62) (167)

$$= -\frac{dw}{\rho_0}\left(\lambda_X\left(\boldsymbol{\mu} - \boldsymbol{\pi}_0 + \mathbf{e}_0\right) + \left(\lambda_P - \lambda_X\right)\hat{\rho}_{\substack{new\\simple}}\boldsymbol{\pi}\right) \tag{168}$$

4.2. Estimation of the price coefficient

Premise 2 established a framework whereby parameters are estimated using OLS. It is well recognized in the literature (for instance Bray (1982) and Routledge (1999)) that estimation of the price regression coefficient using OLS is misspecified because price is non-stationary. In the context of this model the problem is that the price vector moves around in coefficient space. If the price coefficient is estimated as one element of a multiple regression then it will be unstable although the movement in price is only small. It will be shown below that because of estimation error, profits are difficult in the first period after estimation and investors are likely to maintain estimates over several periods. Estimates which are unstable will not be useful for this purpose so investors will be obliged to consider this problem of multicollinearity.

Methodologically, multiple regression has been assumed (in Premise 2) neither because it is indicated *a priori* nor because it is standard market practice. Rather, it is a method of processing data which is known to be generally effective. As analysis has revealed a statistical flaw (unstable regressor combined with multicollinearity) which makes it less suitable for this particular purpose, the assumption must be refined. We suppose:

PREMISE 10: ORTHOGONAL PRICE REGRESSOR. Investors remove multicollinearity from the data by creating data variables which are orthogonal to price before carrying out multiple regression.

Orthogonal variables in data space imply orthogonal variables in coefficient space.

$$\mathbf{x}_\perp = \left(\mathbf{I} - \mathbf{p}\left(\mathbf{p}'\mathbf{p}\right)^{-1}\mathbf{p}'\right)\mathbf{x} = \left(\mathbf{I} - \mathbf{X}\boldsymbol{\pi}\left(\boldsymbol{\pi}'\mathbf{X}'\mathbf{X}\boldsymbol{\pi}\right)^{-1}\boldsymbol{\pi}'\mathbf{X}'\right)\mathbf{X}\mathbf{a} = \mathbf{X}\left(\mathbf{I} - \boldsymbol{\pi}\left(\boldsymbol{\pi}'\boldsymbol{\pi}\right)^{-1}\boldsymbol{\pi}'\right)\mathbf{a} = \mathbf{X}\mathbf{a}_\perp \tag{169}$$

Premise 10 removes the instability problem for the price coefficient. When an explanatory variable is orthogonal to all other explanatory variables in an equation then the multiple regression coefficient is identical to the simple regression coefficient. So we take it that:

$$\hat{\rho}_{\substack{new\\multiple}} = \hat{\rho}_{\substack{new\\simple}} \tag{170}$$

and ‘simple’ is dropped as a subscript.

4.3. Negative price coefficient

For the purposes of the analysis the following working variables will be employed:

- y is the component of the price $\boldsymbol{\pi}$ in the direction of return $\boldsymbol{\mu}$, 'collinear price'.
- x is the remaining component of price, 'orthogonal price'. All of the individual $N-1$ orthogonal components in N dimensional coefficient space are added together.

RESULT 28. *The price regression coefficient is given asymptotically by*

$$\hat{\rho}_{new} = y^{-1} - 1 - x^2 + e_P \tag{171}$$

where $$\boldsymbol{\pi} = \begin{bmatrix} x \\ y \end{bmatrix}, \boldsymbol{\mu} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \tag{172}$$

$$\mathbf{e} = \begin{bmatrix} e_x \\ e_y \end{bmatrix}, e_P = \frac{xe_x + ye_y}{x^2 + y^2} \tag{173}$$

PROOF: Denote $E\left[\hat{\rho}_{new}\right]$ as $\overline{\rho}_{new}$.

$$\overline{\rho}_{new} = E\left[\left(\boldsymbol{\pi}'\boldsymbol{\pi}\right)^{-1} \boldsymbol{\pi}'\left(\boldsymbol{\mu} - \boldsymbol{\pi} + \mathbf{e}\right)\right] = \frac{(x,y)\cdot(-x,1-y)}{(x,y)\cdot(x,y)} \text{ noting } \mathbf{e} \text{ goes out} \tag{174}$$

$$= \frac{y}{x^2 + y^2} - 1 \tag{175}$$

$$\frac{\partial \overline{\rho}_{new}}{\partial x} = -2xy\left(x^2 + y^2\right)^{-2} \tag{176}$$

$$\left.\frac{\partial \overline{\rho}_{new}}{\partial x}\right|_{y=1} = -2x\left(x^2 + 1\right)^{-2} \approx -2x \quad \text{asymptotically} \tag{177}$$

$$\frac{\partial \overline{\rho}_{new}}{\partial y} = \left(x^2 - y^2\right)\left(x^2 + y^2\right)^{-2} \tag{178}$$

$$\left.\frac{\partial \overline{\rho}_{new}}{\partial y}\right|_{x=0} = -y^{-2} \tag{179}$$

$$\overline{\rho}_{new}\Big|_{\substack{x=X \\ y=Y}} = \overline{\rho}_{new}\Big|_{\substack{x=0 \\ y=1}} + \int_{\substack{x=0 \\ y=1}}^{X} \frac{\partial \overline{\rho}_{new}}{\partial x} dx + \int_{\substack{y=1 \\ x=X}}^{Y} \frac{\partial \overline{\rho}_{new}}{\partial y} dy \tag{180}$$

$$\approx 0 + \int_{\substack{x=0 \\ y=1}}^{X} -2x \, dx + \int_{\substack{y=1 \\ x=0}}^{Y} \left(-y^{-2}\right) dy \tag{181}$$

$$= -x^2 + \left(\frac{1}{y} - 1\right) \quad \# \tag{182}$$

so we put this together with the even distribution version of the price change equation (165) to arrive at the following representation of market dynamics:

$$\begin{bmatrix} x_1 \\ y_1 \end{bmatrix} = \begin{bmatrix} x_0 \\ y_0 \end{bmatrix} - \frac{dw}{\rho_0}\left[\lambda_X \begin{bmatrix} 0 - x_0 + e_x \\ 1 - y_0 + e_y \end{bmatrix} + \left(\lambda_P - \lambda_X\right)\left(y_0^{-1} - 1 - x_0^2 + e_P\right)\begin{bmatrix} x_0 \\ y_0 \end{bmatrix}\right] \tag{183}$$

$$\rho_1 = b_{price}\left(1 - dw\right)\hat{\rho}_0 + b_{price} dw \cdot \hat{\rho}_{new} \qquad \text{(59) restated} \tag{184}$$

This system is not linear. For the purposes of this paper we will obtain rough solutions of the expectations using approximations. If we take expectations of the *y* coordinate we get:

$$0=\lambda_P E\left[\frac{1}{\rho}\right]-\lambda_P E\left[\frac{y}{\rho}\right]-\left(\lambda_P-\lambda_X\right)E\left[\frac{x^2 y}{\rho}\right] \tag{185}$$

The presence of the price coefficient ρ as denominator means this expression does not lend itself readily to finding $E\left[y\right]$. Observe however that the rate of adjustment of the price coefficient ρ is $b_{price}dw$, and of the *x*,*y* coordinates is $-\frac{dw}{\rho}$. The rate of adjustment of *x*, *y* is larger by two orders of magnitude than the rate of adjustment of ρ, while the variables *x*, $1-y$ are around the same order of magnitude as ρ. It appears that variation in *x, y* is on a different scale to variation in ρ - the price coordinates *x* and *y* will adjust to equilibrium while the price coefficient ρ essentially remains constant. We restate the independence property in this context:

Practical independence of price coefficient: The price coordinates *x* and *y* can be treated as independent of the price coefficient ρ.

RESULT 29. *The expected orthogonal price is zero:*

$$\bar{x}=0 \tag{186}$$

where $\bar{x}$ *denotes the expected value of x.*

PROOF: The process

$$x_1=x_0-\frac{dw}{\rho_0}\lambda_X\left(-x_0+e_x\right)+\left(\lambda_P-\lambda_X\right)\left(y_0^{-1}-1-x_0^2+e_P\right)x_0 \tag{187}$$

defines a Markov chain which converges to a stable distribution $f\left(x\right)$. Consider a process starting from the opposite starting point $-x_0$:

$$x_1\Big|_{\substack{x=-x_0\\ e=-e_x}}=\left(-x_0\right)-\frac{dw}{\rho_0}\lambda_X\left(x_0-e_x\right)-\left(\lambda_P-\lambda_X\right)\left(y_0^{-1}-1-x_0^2+e_P\right)x_0 \tag{188}$$

$$=-x_1 \tag{189}$$

Given the probability of e_x equals the probability of $-e_x$, this process mirrors the original distribution at every point and converges to distribution $g\left(x\right)$ such that $g\left(-x\right)=f\left(x\right)$. But the two processes are identical although starting points are opposite, so the two processes will converge to the same stable distribution: $g\left(x\right)=f\left(x\right)$. Therefore

$$f\left(x\right)=f\left(-x\right) \qquad \# \tag{190}$$

RESULT 30. *The expected value of collinear price* $\bar{y}$ *and the expected value of the price coefficient* $\bar{\rho}$ *are given by:*

$$\bar{y}=1-\frac{\lambda_P-\lambda_X}{\lambda_P}E\left[x^2 y\right] \tag{191}$$

$$\bar{\rho}\approx b_{price}\left(\frac{\lambda_P-\lambda_X}{\lambda_P}E\left[x^2 y\right]-E\left[x^2\right]\right) \tag{192}$$

PROOF: Rearrange (185) to get:

$$E\left[\frac{y}{\rho}\right]=E\left[\frac{1}{\rho}\right]-\frac{\lambda_P-\lambda_X}{\lambda_P}\cdot\frac{E\left[\frac{x^2y}{\rho}\right]}{E\left[\frac{1}{\rho}\right]} \tag{193}$$

and result for $\bar{y}$ follows from independence of price coefficient ρ.

Now $y^{-1}-1=\frac{1-y}{y}\approx 1-y$ given y is close to one. (194)

so $\hat{\rho}_{new}\approx 1-y-x^2+e_P$ applying (194) to (171) (195)

Then $\bar{\hat{\rho}}=E\left[\sum_t weight_t\cdot\hat{\rho}^t_{new}\right]$ (196)

$\approx\sum_t weight_t\cdot E\left[1-y_t-x_t^2+e_{Pt}\right]$ given $weight_t$ is constant (197)

$=1-\bar{y}-\text{var}\,x$ given $\sum_t weight_t=1$ (198)

$=\frac{\lambda_P-\lambda_X}{\lambda_P}E\left[x^2y\right]-E\left[x^2\right]$ by (191) (199)

$\bar{\rho}=b_{price}\bar{\hat{\rho}}$ by (61) # (200)

Let var x denote the variance of x around the origin, $E\left[x^2\right]$. If as we expect, coordinates x and y are not strongly dependent, then the results can be further simplified to

$\bar{y}\approx 1-\frac{\lambda_P-\lambda_X}{\lambda_P}E\left[x^2\right]\cdot E\left[y\right]\approx 1-\frac{\lambda_P-\lambda_X}{\lambda_P}\text{var}\,x$ noting $\bar{y}\approx 1$ (201)

$\bar{\rho}\approx b_{price}\left(\frac{\lambda_P-\lambda_X}{\lambda_P}E\left[x^2\right]\cdot E\left[y\right]-E\left[x^2\right]\right)\approx-\frac{b_{price}\lambda_X}{\lambda_P}\text{var}\,x$ similarly (202)

The objective model satisfies the fundamental requirement that the price coefficient be negative. This characteristic is intrinsic to the model and Assumption 1 is not required.

4.4. *Exogenous price coefficient and stability formulae*

The result (202) does not present the price coefficient in terms of exogenous variables. To do this we need an estimate of the variance of price coordinate x. From (183) the equation for x is:

$$x_1=x_0-\frac{dw}{\rho_0}\left(\lambda_X\left(-x_0+e_x\right)+\lambda_P\left(y_0^{-1}-1-x_0^2+e_P\right)x_0\right) \tag{203}$$

For these purposes we ignore the second term $\lambda_P\hat{\rho}_{new}x_0$ because $\hat{\rho}_{new}$ can be positive or negative and its impact on x will tend to cancel over time. Proceeding on this basis we can derive:

RESULT 31. *An approximate expression for the variance for the orthogonal price coordinate is:*

$$\operatorname{var} x \approx \frac{a}{2}\sigma_X^2 \tag{204}$$

where $\sigma_X^2 = \frac{N-1}{N}\cdot\sigma^2 \approx \sigma^2$ *is the variance of the error term* e_x (205)

and $a = -\frac{dw\cdot\lambda_X}{\bar{\rho}}$ (206)

PROOF: Pursuant to the above we modify (203) to get:

$$x_1 = x_0 - \frac{dw}{\rho_0}\lambda_X\left(-x_0 + e_x\right) = x_0\left(1-\left(-\frac{dw}{\rho_0}\lambda_X\right)\right) + \left(-\frac{dw}{\rho_0}\lambda_X\right)e_x \tag{207}$$

Introduce the further approximation that

$$E\left[\rho^2\right] = \bar{\rho}^2 + \operatorname{var}\rho \approx \bar{\rho}^2 \tag{208}$$

Then we can square both sides and take expectations to get:

$$\operatorname{var} x = \operatorname{var} x\left(1-a\right)^2 + a^2\sigma_X^2 \tag{209}$$

$$\operatorname{var} x\left(1-\left(1-a\right)^2\right) = a^2\sigma_X^2 \tag{210}$$

$$\operatorname{var} x = \frac{a}{2-a}\sigma_X^2 \approx \frac{a}{2}\sigma_X^2 \quad \text{given a is small relative to 2.} \qquad \# \tag{211}$$

Armed with this admittedly rough and ready approximation we can derive an expression for $\bar{\rho}$ in terms of exogenous variables. The reader may like to remember that the key theoretical result, that the price coefficient ρ is negative, is already established at (199) and does not rely on this concatenation of inexactitudes.

Notice that the variance is directly proportional to update proportion *dw*. One might expect that more updating would lead to a more precise variable *x*, but in fact it amplifies errors. When the update proportion is small errors tend to cancel out and *x* moves slowly but steadily to the central position.

RESULT 32: EXOGENOUS PRICE COEFFICIENT FORMULA.

$$\bar{\rho} = -\lambda_X\sigma_X\sqrt{\frac{b_{price}\cdot dw}{2\lambda_P}} \tag{212}$$

PROOF: $\operatorname{var} x = \frac{\left(-\frac{dw\cdot\lambda_X}{\bar{\rho}}\right)}{2}\sigma_X^2$ from (204),(206) (213)

$\operatorname{var} x = -\frac{\bar{\rho}\,\lambda_P}{b_{price}\lambda_X}$ from (202), eliminating var *x* yields the result.# (214)

Finally a prima facie stability condition can be imposed on the update proportion dw.

RESULT 33: EXOGENOUS STABILITY CONDITION. *For stability we require that the update proportion is constrained according to:*

$$dw < \frac{\sigma_X^2}{2}\left(\frac{b_{price}}{\lambda_P}\right)\left(\frac{\lambda_X}{\lambda_P}\right)^2 \tag{215}$$

PROOF: We apply the stability condition (125) using

$$\lambda_H^{\max} = \lambda_P \tag{216}$$

to get $\bar{\rho} < -dw \cdot \lambda_P$ (217)

Substituting for $\bar{\rho}$ using (212) yields the result. #

The importance of this result lies in the implication that frequent reestimation is inconsistent with market stability. A bubble environment is precisely where one might expect fervent reestimation, and this is one possible explanation of the mechanism which sustains a bubble. The result is consistent with the Brock Hommes (1997) emphasis on 'intensity of choice' as the key parameter determining market stability.

4.5. Simulation

As the results depend on approximations they were tested by simulation. The system implemented was

$$\begin{bmatrix} x_1 \\ y_1 \end{bmatrix} = \begin{bmatrix} x_0 \\ y_0 \end{bmatrix} - \frac{dw}{\rho_0}\left[\lambda_X \begin{bmatrix} 0 - x_0 + e_x \\ 1 - y_0 + e_y \end{bmatrix} + \left(\lambda_P - \lambda_X\right)\hat{\rho}_{new}\begin{bmatrix} x_0 \\ y_0 \end{bmatrix}\right] \tag{218}$$

$$\rho_1 = b_{price}\left(1 - dw\right)\hat{\rho}_0 + b_{price} dw \cdot \hat{\rho}_{new} \qquad \text{(59) restated} \tag{219}$$

Error e_x, e_y is normally distributed. The price regression coefficient is calculated exactly; in this way any inaccuracies introduced by the approximation for $\hat{\rho}_{new}$ are avoided.

$$\hat{\rho}_{new} = \left(\boldsymbol{\pi}'\boldsymbol{\pi}\right)^{-1}\boldsymbol{\pi}'\left(\boldsymbol{\mu} - \boldsymbol{\pi} + \mathbf{e}\right) = \frac{\left(x, y\right)\cdot\left(-x + e_x, 1 - y + e_y\right)}{\left(x, y\right)\cdot\left(x, y\right)} \tag{220}$$

The ratio of errors r_{error} is the ratio of the standard deviation of collinear error to the standard deviation of orthogonal error.

$$r_{error} = \frac{\sigma_y}{\sigma_x} \tag{221}$$

Parameter	Symbol	Value
Eigenvalue of non-price eigenvectors of estimation matrix **H**	λ_X	0.5
Eigenvalue of the price eigenvector of estimation matrix **H**	λ_P	0.7
Proportion of investors using a price regressor	b_{price}	0.5
Initial value of orthogonal price	x_0	0
Initial value of collinear price	y_0	1

TABLE 2: Shows the parameters of the price coefficient simulation.

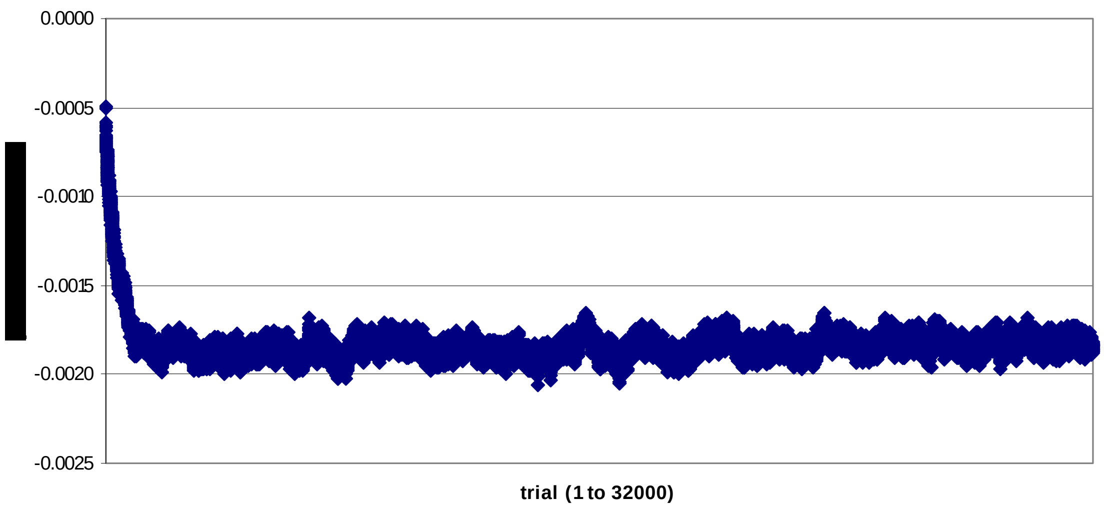
Results:

- FIG. 3. The graphs depict the series for the price coefficient ρ and the collinear component of price y. They are generated using typical settings for the parameters:

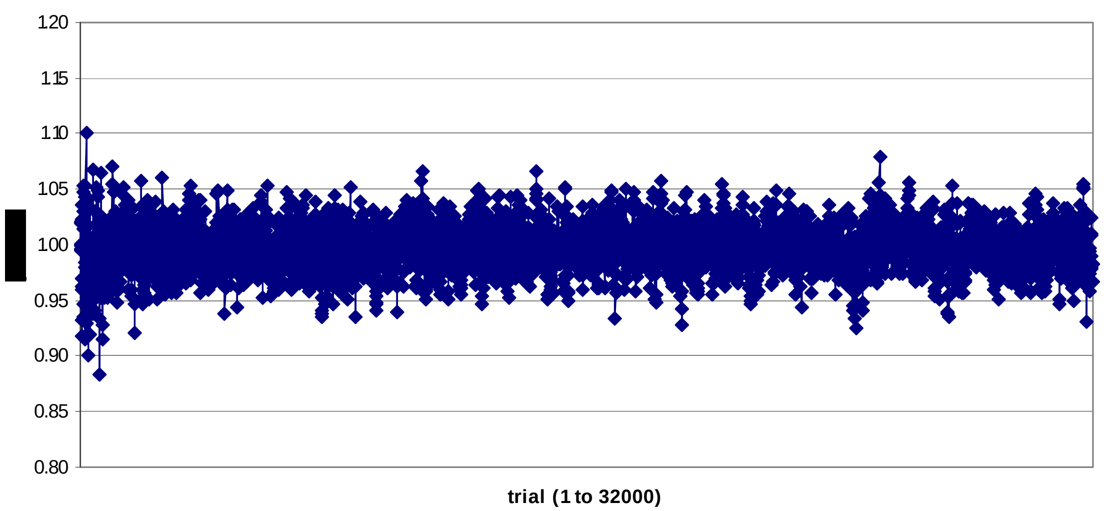
- o dw, the proportion of investors updating estimates in each period, is 0.001
- o σ_x, standard deviation of orthogonal error, is 0.20
- o σ_y, standard deviation of collinear error, is 0.04. (r_{error} =0.20)

The simulation starts from the small value for ρ_0 of 0.0005. The variables converge to the expected values despite the initial disturbance. As expected, convergence of the price coefficient is several magnitudes slower (950 iterations) than that of the other variables (less than five iterations). The logic underlying Assumption 2 is validated.

Price coefficient ρ:



Collinear price *y*:



For reasons of space the graph of orthogonal price *x* is omitted: it is visually similar to collinear price and shows variations around a mean of zero as expected.

- Independence assumptions: The variables were tested for independence by calculating the following ratio, which is 100% for independent data

$$r_{independence} = \frac{E[ab]}{E[a] \cdot E[b]} \quad (222)$$

because this is the property which is relevant to the assumptions made in the derivations. Results for the standard data $dw = 0.001, \sigma_x = 0.20, \sigma_y = 0.04$:

- For y and ρ, $r_{independence} = 100.00\%$
- For x^2 and ρ, $r_{independence} = 101.15\%$
- For x^2 and y, $r_{independence} = 100.23\%$

On this basis the assumption of independence is upheld. Similar results held at all test points.

- Accuracy of the approximation of the price regression coefficient $\hat{\rho}_{new}$: the approximation given at (171) was tested along with two other approximations. Each variant was regressed on the true price regression coefficient and the slope and coefficient of determination R^2 recorded. Each of the approximations is excellent, even though values of the x variable of up to 0.3 are recorded. The most accurate version outperforms the others but $\hat{\rho}_{new} \approx y^{-1} - 1 - x^2$ is preferred for its analytic tractability.

$\hat{\rho}_{new}$ approximation	Comment	Regression slope $\hat{\beta}$	R^2
$y^{-1} - 1 - x^2$	This is variant used	1.0068	.99973
$1 - y - x^2$	Simplest version	1.0062	.99971
$y^{-1} + \left(x^2 + 1\right)^{-1} - 2$	Most accurate version	1.0028	.99992

TABLE 3: Shows the accuracy of three approximate expressions for the price regression coefficient $\hat{\rho}_{new}$.

- Accuracy of the price coefficient formula (212) is measured by $r_{accuracy}$, the ratio of the simulation value to the formula value. $r_{accuracy}$ is measured over a wide range of parameter values. The theoretical price coefficient is surprisingly accurate considering the approximations used to derive it. When the ratio of errors r_{error} and the update proportion dw are low, these approximations are more nearly satisfied and this is borne out in the results.

$r_{accuracy}$ (%)					
Orthogonal error s.d. σ_x *(stability limit)*	Update proportion *dw*	Ratio of errors $r_{error} = \sigma_y / \sigma_x$			
		0.00	0.05	0.20	0.50
0.20 *(dw < 0.0075)*	0.000001	100.2	99.7	90.2	33.7
	0.00001	100.3	100.0	93.9	41.9
	0.0001	102.0	101.6	95.4	**unstable**
	0.001	105.9	105.5	99.2	**unstable**
	0.01	119.5	119.2	111.6	unstable

	0.02	131.5	131.4	unstable	unstable
	0.03	unstable	unstable	unstable	unstable
0.01(*dw < 0.000018*)	0.0000001	101.0	101.0	94.0	38.1
	0.000001	104.5	104.0	99.5	**unstable**
	0.00001	114.2	113.3	106.8	**unstable**
	0.0001	unstable	unstable	unstable	unstable

TABLE 4: Shows $r_{accuracy}$, the ratio of the average simulation price coefficient to the theoretical price coefficient given at (212). If the price coefficient violates the negativity constraint or jumps around, it is described as unstable. Entries above the dashed line are expected to be stable and those below are expected to be unstable. Bolded entries are unstable contrary to the theoretical expectation (215).

- Stability. When the ratio of errors is not large, i.e.

$$r_{error} \le 20\% \qquad (223)$$

the stability condition (215) is essentially correct but for higher ratios instability sets in at much lower values of the update proportion dw. Presumably when the variance of collinear price is high it overwhelms the calming effect of orthogonal data on the price regression coefficient. The stability characteristics of the model are not fully captured by the theory because the derivation of the stability formula does not consider collinear variance. The variance of collinear price will be $\frac{1}{N}$ of total price variance on average so it will be relatively high when the data set contains only a few variables.

5. The economics of the objective model

5.1. Premises and notation

We look at the return to the investor and its implications for the economic consistency of the objective model. The two questions arising are:
- Can the investors' expectations of positive return can be reconciled with an aggregate profit of zero?
- In addition to a negative price coefficient, the price making process requires that price equation (22) has a numerator. Is there an economic incentive for investors to consider non-price data – in other words does this model avoid the Grossman Stiglitz paradox?

Profit is the product of quantity and return. We sum the profit of every observation period in an estimation period by taking the inner product, and use relative b_j rather than absolute weight of money B_j to arrive at the following representation of profit Π_{1*1} within coefficient space.

$$\Pi_j = \mathbf{q}_j{}' \cdot \mathbf{r}_j = b_j \hat{\mathbf{r}}_j \mathbf{r} \qquad (224)$$

where the expected return $\hat{\mathbf{r}}_{T*1}$ of investor *j* who created their estimates in period *t* is given by:

$$\hat{\mathbf{r}}_j = \mathbf{X}_1 \boldsymbol{\alpha}_{jt} \boldsymbol{\alpha}_{jt}' \left(\boldsymbol{\mu} - \boldsymbol{\pi}_t + \mathbf{e}_t \right) \quad \text{by (33),(34)} \quad (225)$$

So $$\Pi_j = b_j \left(\mathbf{X}_1 \boldsymbol{\alpha}_{jt} \boldsymbol{\alpha}_{jt}' \left(\boldsymbol{\mu} - \boldsymbol{\pi}_t + \mathbf{e}_t \right) \right)' \left(\mathbf{X}_1 \boldsymbol{\mu} - \mathbf{X}_1 \boldsymbol{\pi} + \mathbf{u}_1 \right) \quad \text{by (28), (225)} \quad (226)$$

$$= b_j \left(\boldsymbol{\mu} - \boldsymbol{\pi}_t + \mathbf{e}_t \right)' \boldsymbol{\alpha}_{jt} \boldsymbol{\alpha}_{jt}' \left(\boldsymbol{\mu} - \boldsymbol{\pi}_1 + \mathbf{e}_1 \right) \quad \text{by (3), (35)} \quad (227)$$

As price has been assumed to be orthogonal to the other regressors the regression matrix of investor *j* can be written as:

$$\tilde{\mathbf{a}}_{j0} \left(\tilde{\mathbf{a}}_{j0}' \tilde{\mathbf{a}}_{j0} \right) \tilde{\mathbf{a}}_{j0}' = \mathbf{a}_{j0} \left(\mathbf{a}_{j0}' \mathbf{a}_{j0} \right) \mathbf{a}_{j0}' + \boldsymbol{\pi}_0 \left(\boldsymbol{\pi}_0' \boldsymbol{\pi}_0 \right) \boldsymbol{\pi}_0' \quad (228)$$

The two sets of regression matrices – data and price - can be expressed in terms of orthogonal unit vectors $\mathbf{v}_{jt}, \mathbf{u}_t$ using the argument given earlier for $\boldsymbol{\alpha}_{jt}$. Since $\mathbf{v}_{jt}$ is a basis for **a**, it is orthogonal to **u**. ($\mathbf{a}'\boldsymbol{\pi} = \mathbf{0}$ so $\left(\mathbf{v}'\mathbf{v}\right)^{-0.5} \mathbf{v}' \cdot \mathbf{u} \left(\mathbf{u}'\mathbf{u}\right)^{-0.5} = \mathbf{0}$). Rewriting:

$$\boldsymbol{\alpha}_{jt} \boldsymbol{\alpha}_{jt}' = \mathbf{v}_{jt} \mathbf{v}_{jt}' + \mathbf{u}_t \mathbf{u}_t' \quad \text{(228) rewritten} \quad (229)$$

There is a difference in the profit expectation of *new* investors, and investors who estimated a while ago. *Mature* investors are defined as those whose estimation period *t* is sufficiently removed from the present that current price $\boldsymbol{\pi}_1$ is not correlated with the price or error $-\boldsymbol{\pi}_t + \mathbf{e}_t$. The set of *old* investors is a broader set which includes investors who last estimated in estimation period -1 and in general does not satisfy the no-correlation criterion.

The expected value of price implied by (191) is denoted $\overline{\boldsymbol{\pi}}$. For the purposes of this section the estimation matrix **H** will be taken as constant.

RESULT 34: *Aggregate profit is zero.*

PROOF: $$\sum_j \Pi_j = \sum_j \mathbf{q}_j' \mathbf{r} = \mathbf{0}_{1*T} \mathbf{r} = 0 \quad \text{by (16), } \mathbf{r} \text{ constant across } j \quad \# \quad (230)$$

5.2. Mature profit

RESULT 35: *For mature investors using price:*

$$E\left[\Pi_j^{mature}\right] = b_j \cdot |\boldsymbol{\mu}| \cdot |\boldsymbol{\mu} - \overline{\boldsymbol{\pi}}| \cdot E\left[R_j^2\right] + b_j \left(\boldsymbol{\mu}' \cdot E\left[\mathbf{u}_{jt} \mathbf{u}_{jt}' \right] - \overline{\boldsymbol{\pi}} \right) \left(\boldsymbol{\mu} - \overline{\boldsymbol{\pi}} \right) \quad (231)$$

where R_j^2 *is the coefficient of determination from regression of return* $\boldsymbol{\mu}$ *onto investor j's data.*

PROOF: $$E\left[\Pi_j^{mature}\right] / b_j = E\left[\left(\boldsymbol{\mu} - \boldsymbol{\pi}_t + \mathbf{e}_t \right)' \boldsymbol{\alpha}_{jt} \boldsymbol{\alpha}_{jt}' \left(\boldsymbol{\mu} - \boldsymbol{\pi}_1 + \mathbf{e}_1 \right) \right] \quad (232)$$

$$= E\left[\left(\boldsymbol{\mu} - \boldsymbol{\pi}_t + \mathbf{e}_t \right)' \boldsymbol{\alpha}_{jt} \boldsymbol{\alpha}_{jt}' \right] E\left[\left(\boldsymbol{\mu} - \boldsymbol{\pi}_1 + \mathbf{e}_1 \right) \right] \quad \text{by definition } \textit{mature} \quad (233)$$

$$= E\left[\left(\boldsymbol{\mu} - \boldsymbol{\pi}_t + \mathbf{e}_t\right)'\left(\mathbf{v}_{jt}\mathbf{v}_{jt}' + \mathbf{u}_t\mathbf{u}_t'\right)\right]\left(\boldsymbol{\mu} - \overline{\boldsymbol{\pi}}\right) \tag{234}$$

$$= E\left[\boldsymbol{\mu}'\mathbf{v}_{jt}\mathbf{v}_{jt}'\left(\boldsymbol{\mu} - \overline{\boldsymbol{\pi}}\right)\right] + \left(\boldsymbol{\mu}' \cdot E\left[\mathbf{u}_{jt}\mathbf{u}_{jt}'\right] - \overline{\boldsymbol{\pi}}'\right)\left(\boldsymbol{\mu} - \overline{\boldsymbol{\pi}}\right) \text{ using orthogonality} \tag{235}$$

$$E\left[\Pi_j^{mature\ data}\right]/b_j = E\left[\boldsymbol{\mu}'\mathbf{v}_{jt}\mathbf{v}_{jt}'\boldsymbol{\mu}\right]\frac{\left|\boldsymbol{\mu} - \overline{\boldsymbol{\pi}}\right|}{\left|\boldsymbol{\mu}\right|} \tag{236}$$

$$= E\left[R_j^2\right] \cdot \left|\boldsymbol{\mu}\right|^2 \cdot \frac{\left|\boldsymbol{\mu} - \overline{\boldsymbol{\pi}}\right|}{\left|\boldsymbol{\mu}\right|} \qquad \text{noting } R_j^2 = \frac{\text{ESS}}{\text{TSS}} = \frac{\left|\hat{\boldsymbol{\mu}}\right|^2}{\left|\boldsymbol{\mu}\right|^2} \quad \# \tag{237}$$

It might be asked how the expectation of non-price variables $E\left[\boldsymbol{\mu}'\mathbf{v}_{jt}\mathbf{v}_{jt}'\boldsymbol{\mu}\right]$ can capture $\boldsymbol{\mu}$, when the data variables $\mathbf{v}_{jt}$ are orthogonal to price which has expected value $\overline{\boldsymbol{\pi}} = c \cdot \boldsymbol{\mu}$. The answer is that the expectation of vectors orthogonal to price is not the same thing as vectors orthogonal to the expectation of price. $E\left[\mathbf{v}_{jt}\mathbf{v}_{jt}\right]$ does have components in the direction of $\boldsymbol{\mu}$.

We now analyze the expression $\boldsymbol{\mu}' \cdot E\left[\mathbf{u}_{jt}\mathbf{u}_{jt}'\right] - \overline{\boldsymbol{\pi}}'$ using the working variables x, y defined in Section 4 above and the same distribution assumptions.

LEMMA 36: *If return is projected onto price as per the expression* $\boldsymbol{\mu}' \cdot \mathbf{u}_{jt}\mathbf{u}_{jt}'$ *in* (231), *the expectation of predicted return is given asymptotically by:*

$$\mathrm{E}\begin{bmatrix}\hat{\mu}_x \\ \hat{\mu}_y\end{bmatrix} \approx \begin{bmatrix} 0 \\ 1 - \operatorname{var} x\left(1 + 2\dfrac{\lambda_P - \lambda_X}{\lambda_P} \cdot \operatorname{var} x - \dfrac{\lambda_P}{\lambda_X} \cdot \dfrac{\operatorname{var} x}{N-1}\right)\end{bmatrix} \tag{238}$$

where $\hat{\mu}_x$ *denotes the orthogonal component of predicted return* $\hat{\boldsymbol{\mu}}$ *and* $\hat{\mu}_y$ *denotes the collinear component.*

PROOF: $\begin{bmatrix}\hat{\mu}_x \\ \hat{\mu}_y\end{bmatrix} = \begin{bmatrix}x \\ y\end{bmatrix}\left(\begin{bmatrix}x & y\end{bmatrix}\begin{bmatrix}x \\ y\end{bmatrix}\right)^{-1}\begin{bmatrix}x & y\end{bmatrix}\begin{bmatrix}0 \\ 1\end{bmatrix} = \dfrac{1}{x^2 + y^2}\begin{bmatrix}xy \\ y^2\end{bmatrix}$ (239)

Consider the x component. Take expectations of (187)

$$E[x] = E[x] + E\left[\frac{dw}{\rho_0}\right]\lambda_X E[x] + \left(\lambda_P - \lambda_X\right)E\left[\left(1 - y - x^2\right)x\right] \tag{240}$$

using $1 - y$ in place of $y^{-1} - 1$. Since $E[x] = 0$, $E\left[x^3\right] = 0$ we get:

$$E[xy] = 0 \tag{241}$$

Asymptotically $x^2 + y^2$ goes to unity, so $E\left[\hat{\mu}_x\right] = E\left[\dfrac{xy}{x^2 + y^2}\right] = 0$ asymptotically.

Turning to the y component,

$$E\left[\hat{\mu}_y\right] = E\left[\frac{1}{\frac{x^2}{y^2}+1}\right] \quad (242)$$

$$\geq E\left[1-\frac{x^2}{y^2}\right] \qquad \text{but asymptotically equal} \quad (243)$$

$$\approx 1-\frac{\operatorname{var} x}{\bar{y}^2+\operatorname{var} y} \qquad \mathbf{\Sigma} \text{ is diagonal by (141) so } x,y \text{ independent} \quad (244)$$

$$\leq 1-\operatorname{var} x\left(1-\left(\bar{y}^2-1\right)-\operatorname{var} y\right) \qquad \text{but asymptotically equal} \quad (245)$$

$$= 1-\operatorname{var} x\left(1+2\frac{\lambda_P-\lambda_X}{\lambda_P}\cdot\operatorname{var} x-\frac{\lambda_P}{\lambda_X}\cdot\frac{\operatorname{var} x}{N-1}\right) \qquad \text{using the following:} \quad (246)$$

$$\bar{y}^2 \approx \left(1-\frac{\lambda_P-\lambda_X}{\lambda_P}\cdot\operatorname{var} x\right)^2 \approx 1-2\cdot\frac{\lambda_P-\lambda_X}{\lambda_P}\cdot\operatorname{var} x \qquad \text{by (201)} \quad (247)$$

and $$\operatorname{var} y = \frac{\lambda_P}{\lambda_X}\cdot\frac{\operatorname{var} x}{N-1} \qquad \text{by (141), } x \text{ incorporates } N-1 \text{ degrees of freedom} \quad (248)$$

RESULT 37. *The price component of mature profit is negative if:*

$$\operatorname{var} x < 1 \quad (249)$$

and $$\lambda_X > \frac{\lambda_P}{N-1} \quad (250)$$

PROOF: $$E\left[\Pi_j^{mature\ price}\right] = b_j\left(\boldsymbol{\mu}'\cdot E\left[\mathbf{u}_{jt}\mathbf{u}_{jt}'\right]-\bar{\boldsymbol{\pi}}'\right)\left(\boldsymbol{\mu}-\bar{\boldsymbol{\pi}}\right) \quad (251)$$

$$= b_j\left(\left(1-\operatorname{var} x\left(1+2\frac{\lambda_P-\lambda_X}{\lambda_P}\cdot\operatorname{var} x-\frac{\lambda_P}{\lambda_X}\cdot\frac{\operatorname{var} x}{N-1}\right)\right)-\left(1-\frac{\lambda_P-\lambda_X}{\lambda_P}\operatorname{var} x\right)\right)\boldsymbol{\mu}'\left(\boldsymbol{\mu}-\bar{\boldsymbol{\pi}}\right) \quad (252)$$

substituting (201),(238)

$$= b_j \operatorname{var} x\left(-\frac{\lambda_X}{\lambda_P}-2\cdot\frac{\lambda_P-\lambda_X}{\lambda_P}\cdot\operatorname{var} x+\frac{\lambda_P}{\lambda_X}\cdot\frac{\operatorname{var} x}{N-1}\right)\boldsymbol{\mu}'\left(\boldsymbol{\mu}-\bar{\boldsymbol{\pi}}\right) \quad (253)$$

Now $$-\frac{\lambda_X}{\lambda_P}-2\cdot\frac{\lambda_P-\lambda_X}{\lambda_P}\cdot\operatorname{var} x+\frac{\lambda_P}{\lambda_X}\cdot\frac{\operatorname{var} x}{N-1} < -\frac{\lambda_X}{\lambda_P}-2\cdot\frac{\lambda_P-\lambda_X}{\lambda_P}\cdot\operatorname{var} x+\operatorname{var} x \ \text{(250)} \quad (254)$$

$$= \frac{\lambda_X}{\lambda_P}\left(\operatorname{var} x-1\right)+\operatorname{var} x\left(\frac{\lambda_X}{\lambda_P}-1\right) \qquad \text{on rearrangement} \quad (255)$$

$$< 0 \qquad \text{noting } \frac{\lambda_X}{\lambda_P}\leq 1,\ \operatorname{var} x < 1 \quad (256)$$

and since $\boldsymbol{\mu}'\left(\boldsymbol{\mu}-\bar{\boldsymbol{\pi}}\right) > 0$, result follows from (253). #

The premises impose minimal constraints and it can be taken that the price component of mature profit is generally negative.

RESULT 38. *For mature investors not using price:*

$$E\left[\Pi_j^{mature}\right]/b_j = \left|\boldsymbol{\mu}-\overline{\boldsymbol{\pi}}\right|^2 E\left[R_j^2\right] \tag{257}$$

PROOF: $$E\left[\Pi_j^{mature}\right]/b_j = E\left[\left(\boldsymbol{\mu}-\boldsymbol{\pi}_t+\mathbf{e}_t\right)'\boldsymbol{\alpha}_j\boldsymbol{\alpha}_j'\left(\boldsymbol{\mu}-\boldsymbol{\pi}_1+\mathbf{e}_1\right)\right] \tag{258}$$

$$= E\left[\left(\boldsymbol{\mu}-\boldsymbol{\pi}_t\right)'\right]\mathbf{v}_j\mathbf{v}_j'E\left[\left(\boldsymbol{\mu}-\boldsymbol{\pi}_1\right)\right] \quad \text{as before} \tag{259}$$

$$= \left|\boldsymbol{\mu}-\overline{\boldsymbol{\pi}}\right|^2 E\left[R_j^2\right] \quad \text{in this case } \boldsymbol{\pi}_t \text{ is not orthogonal to } \boldsymbol{\alpha}_j \quad \# \tag{260}$$

5.3. New profit

LEMMA 39. *The expected profit of new investors is given by:*

$$E\left[\Pi_j^{new}\right] = b_j E\left[\sum_k \overline{\mathbf{o}}_{jk}'\left(\overline{\boldsymbol{\mu}-\boldsymbol{\pi}_0}\cdot\overline{\boldsymbol{\mu}-\boldsymbol{\pi}_0}'\dot{\boldsymbol{\lambda}}-\sigma^2\ddot{\boldsymbol{\lambda}}\right)\overline{\mathbf{o}}_{jk}\right] \tag{261}$$

where vincula indicate the variable relative to the orthogonal basis defined by the eigenvectors, derived by multiplying the original form of the variable by $\boldsymbol{\Lambda}'$; thus $\overline{\boldsymbol{\alpha}}_{j0}$ denotes the $\boldsymbol{\alpha}_{j0}$ matrix expressed relative to the eigenvectors, i.e. $\overline{\boldsymbol{\alpha}}_{j0} = \boldsymbol{\Lambda}'\boldsymbol{\alpha}_{j0}$. $\overline{\boldsymbol{\alpha}}_{j0}$ is partitioned into columns $\left[\overline{\mathbf{o}}_{j1} \quad \overline{\mathbf{o}}_{j2} \quad \right]$.

PROOF: $$\Pi_{j1} = b_j\left(\boldsymbol{\mu}-\boldsymbol{\pi}_0+\mathbf{e}_0\right)'\boldsymbol{\alpha}_{j0}\boldsymbol{\alpha}_{j0}'\left(\boldsymbol{\mu}-\boldsymbol{\pi}_1+\mathbf{e}_1\right) \tag{262}$$

$$= b_j\left(\boldsymbol{\mu}-\boldsymbol{\pi}_0+\mathbf{e}_0\right)'\boldsymbol{\alpha}_{j0}\boldsymbol{\alpha}_{j0}'\left(\boldsymbol{\mu}-\boldsymbol{\pi}_0+\mathbf{e}_1+\frac{dw}{\rho_0}\mathbf{H}_\mu\left(\boldsymbol{\mu}-\boldsymbol{\pi}_0+\mathbf{e}_0\right)\right) \text{ using (129)} \tag{263}$$

$$\begin{aligned} &= b_j\left(\boldsymbol{\mu}-\boldsymbol{\pi}_0+\mathbf{e}_0\right)'\boldsymbol{\alpha}_{j0}\boldsymbol{\alpha}_{j0}'\left(\boldsymbol{\mu}-\boldsymbol{\pi}_0+\mathbf{e}_1\right) \\ &\quad + b_j\left(\boldsymbol{\mu}-\boldsymbol{\pi}_0+\mathbf{e}_0\right)'\boldsymbol{\alpha}_{j0}\boldsymbol{\alpha}_{j0}'\frac{dw}{\rho_0}\mathbf{H}_\mu\left(\boldsymbol{\mu}-\boldsymbol{\pi}_0\right) \\ &\quad + b_j\left(\boldsymbol{\mu}-\boldsymbol{\pi}_0+\mathbf{e}_0\right)'\boldsymbol{\alpha}_{j0}\boldsymbol{\alpha}_{j0}'\frac{dw}{\rho_0}\mathbf{H}_\mu\mathbf{e}_0 \end{aligned} \tag{264}$$

$$= \Pi_j^{no\ pricemove} + \Pi_j^{estimation\ pricemove} + \Pi_j^{misestimation\ pricemove} \tag{265}$$

where profit is broken into components corresponding to the terms in (264). Consider the first component which is the profit in the absence of the pricemove from $\boldsymbol{\pi}_0$ to $\boldsymbol{\pi}_1$:

$$E\left[\Pi_j^{no\ pricemove}\right]/b_j = E\left[\left(\boldsymbol{\mu}-\boldsymbol{\pi}_0\right)'\boldsymbol{\alpha}_{j0}\boldsymbol{\alpha}_{j0}'\left(\boldsymbol{\mu}-\boldsymbol{\pi}_0\right)\right] \quad \text{given } \mathbf{e}_0, \mathbf{e}_1 \text{ independent} \tag{266}$$

$$= E\left[\overline{\boldsymbol{\mu}-\boldsymbol{\pi}_0}'\,\overline{\boldsymbol{\alpha}_{j0}}\;\overline{\boldsymbol{\alpha}_{j0}}'\,\overline{\boldsymbol{\mu}-\boldsymbol{\pi}_0}\right] \quad \text{introducing } \boldsymbol{\Lambda}\boldsymbol{\Lambda}' \text{ and multiplying} \tag{267}$$

$$= E\left[\overline{\boldsymbol{\mu}-\boldsymbol{\pi}_0}'\left[\overline{\mathbf{o}}_{j1} \quad \overline{\mathbf{o}}_{j2} \quad \right]\begin{bmatrix}\overline{\mathbf{o}}_{j1}' \\ \overline{\mathbf{o}}_{j2}' \\ \\ \end{bmatrix}\overline{\boldsymbol{\mu}-\boldsymbol{\pi}_0}\right] \tag{268}$$

$$= E\left[\left[\overline{\mathbf{o}_{j1}}'\,\overline{\boldsymbol{\mu}-\boldsymbol{\pi}_0} \quad \overline{\mathbf{o}_{j2}}'\,\overline{\boldsymbol{\mu}-\boldsymbol{\pi}_0} \quad \right]\begin{bmatrix}\overline{\boldsymbol{\mu}-\boldsymbol{\pi}_0}'\,\overline{\mathbf{o}_{j1}} \\ \overline{\boldsymbol{\mu}-\boldsymbol{\pi}_0}'\,\overline{\mathbf{o}_{j2}} \\ \end{bmatrix}\right] \text{ multiply, transpose} \tag{269}$$

$$= E\left[\sum_k \overline{\mathbf{o}_{jk}}'\,\overline{\boldsymbol{\mu}-\boldsymbol{\pi}_0}\cdot\overline{\boldsymbol{\mu}-\boldsymbol{\pi}_0}'\,\overline{\mathbf{o}_{jk}}\right] \qquad \text{multiplying out} \tag{270}$$

The second profit component derives from the adverse movement of price due to correct estimation:

$$E\left[\Pi_j^{estimation\ pricemove}\right]/b_j = E\left[(\boldsymbol{\mu}-\boldsymbol{\pi}_0)'\,\boldsymbol{\alpha}_{j0}\boldsymbol{\alpha}_{j0}'\frac{dw}{\rho_0}\mathbf{H}_\mu(\boldsymbol{\mu}-\boldsymbol{\pi}_0)\right] \mathbf{e}_0 \text{ independent} \tag{271}$$

$$= -E\left[(\boldsymbol{\mu}-\boldsymbol{\pi}_0)'\,\boldsymbol{\Lambda}\boldsymbol{\Lambda}'\boldsymbol{\alpha}_{j0}\boldsymbol{\alpha}_{j0}'\boldsymbol{\Lambda}\ddot{\boldsymbol{\lambda}}\boldsymbol{\Lambda}'(\boldsymbol{\mu}-\boldsymbol{\pi}_0)\right] \text{ substituting } -\boldsymbol{\Lambda}\ddot{\boldsymbol{\lambda}}\boldsymbol{\Lambda}' \text{ for } \frac{dw}{\rho_0}\mathbf{H}_\mu \tag{272}$$

$$= -E\left[\overline{\boldsymbol{\mu}-\boldsymbol{\pi}_0}'\,\overline{\boldsymbol{\alpha}_{j0}}\;\overline{\boldsymbol{\alpha}_{j0}}'\;\ddot{\boldsymbol{\lambda}}\;\overline{\boldsymbol{\mu}-\boldsymbol{\pi}_0}\right] \qquad \text{converting to eigenvector coordinates} \tag{273}$$

$$= -E\left[\sum_k \overline{\mathbf{o}_{jk}}'\,\overline{\boldsymbol{\mu}-\boldsymbol{\pi}_0}\cdot\overline{\boldsymbol{\mu}-\boldsymbol{\pi}_0}'\;\ddot{\boldsymbol{\lambda}}\;\overline{\mathbf{o}_{jk}}\right] \qquad \text{as for the first profit component} \tag{274}$$

The third profit component is the adverse movement of price which is due to error:

$$E\left[\Pi_j^{misestimation\ pricemove}\right]/b_j = E\left[\mathbf{e}_0'\boldsymbol{\alpha}_{j0}\boldsymbol{\alpha}_{j0}'\frac{dw}{\rho_0}\mathbf{H}_\mu\mathbf{e}_0\right] \text{ given } \mathbf{e}_0 \text{ independent} \tag{275}$$

$$= -E\left[\overline{\mathbf{e}_0}'\,\overline{\boldsymbol{\alpha}_{j0}}\;\overline{\boldsymbol{\alpha}_{j0}}'\;\ddot{\boldsymbol{\lambda}}\;\overline{\mathbf{e}_0}\right] \qquad \text{as for second profit component} \tag{276}$$

$$= -E\left[\sum_k \overline{\mathbf{o}_{jk}}'\,\overline{\mathbf{e}_0}\cdot\overline{\mathbf{e}_0}'\;\ddot{\boldsymbol{\lambda}}\;\overline{\mathbf{o}_{jk}}\right] \qquad \text{as for first profit component} \tag{277}$$

$$= -\sum_k \overline{\mathbf{o}_{jk}}'\,\sigma^2\;\ddot{\boldsymbol{\lambda}}\;\overline{\mathbf{o}_{jk}} \qquad \text{as above, see (139)} \quad \# \tag{278}$$

LEMMA 40.

$$E\left[\overline{\boldsymbol{\mu}-\boldsymbol{\pi}_0}\cdot\overline{\boldsymbol{\mu}-\boldsymbol{\pi}_0}'\dot{\boldsymbol{\lambda}} - \sigma^2\ddot{\boldsymbol{\lambda}}\right] = -\boldsymbol{\Sigma} \tag{279}$$

PROOF: $$E\left[\overline{\boldsymbol{\mu}-\boldsymbol{\pi}_0}\cdot\overline{\boldsymbol{\mu}-\boldsymbol{\pi}_0}'\dot{\boldsymbol{\lambda}} - \sigma^2\ddot{\boldsymbol{\lambda}}\right] = \boldsymbol{\Sigma}\dot{\boldsymbol{\lambda}} - \sigma^2\ddot{\boldsymbol{\lambda}} \tag{280}$$

$$= -\sigma^2\left(\mathbf{I}-\dot{\boldsymbol{\lambda}}\right)\left(\mathbf{1}+\dot{\boldsymbol{\lambda}}\right)^{-1} = -\boldsymbol{\Sigma} \qquad \text{using (141)} \quad \# \tag{281}$$

LEMMA 41. An approximate result is:

$$E\left[\Pi_j^{new}\right] \approx -b_j E\left[\sum_k \overline{\mathbf{o}}_{jk}\boldsymbol{\Sigma}\,\overline{\mathbf{o}}_{jk}\right] \tag{282}$$

PROOF: $E\left[\Pi_j^{new}\right]=b_jE\left[\sum_k \overline{\mathbf{o}}_{jk}{}'\left(\overline{\boldsymbol{\mu}-\boldsymbol{\pi}_0}\cdot\overline{\boldsymbol{\mu}-\boldsymbol{\pi}_0}{}'\dot{\boldsymbol{\lambda}}-\sigma^2\ddot{\boldsymbol{\lambda}}\right)\overline{\mathbf{o}}_{jk}\right]$ (283)

$$=b_jE\left[E\left[\sum_k \overline{\mathbf{o}}_{jk}{}'\left(\overline{\boldsymbol{\mu}-\boldsymbol{\pi}_0}\cdot\overline{\boldsymbol{\mu}-\boldsymbol{\pi}_0}{}'\dot{\boldsymbol{\lambda}}-\sigma^2\ddot{\boldsymbol{\lambda}}\right)\overline{\mathbf{o}}_{jk}\middle|\boldsymbol{\alpha}_{j0}\right]\right] \quad (284)$$

$$=b_jE\left[\left[\sum_k \overline{\mathbf{o}}_{jk}{}'E\left[\overline{\boldsymbol{\mu}-\boldsymbol{\pi}_0}\cdot\overline{\boldsymbol{\mu}-\boldsymbol{\pi}_0}{}'\dot{\boldsymbol{\lambda}}-\sigma^2\ddot{\boldsymbol{\lambda}}\middle|\boldsymbol{\alpha}_j\right]\overline{\mathbf{o}}_{jk}\middle|\boldsymbol{\alpha}_{j0}\right]\right] \text{ as } \overline{\mathbf{o}}_{jk} \text{ fixed given } \boldsymbol{\alpha}_{j0} \quad (285)$$

$$\approx -b_jE\left[\sum_k \overline{\mathbf{o}}_{jk}{}'\boldsymbol{\Sigma}\,\overline{\mathbf{o}}_{jk}\right] \qquad \text{by (279)} \qquad \# \quad (286)$$

The rationale for (286) is that the expression $\overline{\boldsymbol{\mu}-\boldsymbol{\pi}_0}\cdot\overline{\boldsymbol{\mu}-\boldsymbol{\pi}_0}{}'\dot{\boldsymbol{\lambda}}-\sigma^2\ddot{\boldsymbol{\lambda}}$ does not depend in any particular way on regressors $\boldsymbol{\alpha}_{j0}$ as they are normalized and embrace all magnitudes of price in a particular direction. The result is good enough for a first order investigation of profits.

RESULT 42. *For new investors using price:*

$$E\left[\Pi_j^{new}\right]=-b_j\left|\boldsymbol{\mu}\right|^2E\left[R_j^2\right]-b_jE\left[\left(\boldsymbol{\mu}-\boldsymbol{\pi}_0\right)'\mathbf{u}_0\mathbf{u}_0{}'\left(\boldsymbol{\mu}-\boldsymbol{\pi}_0\right)\right] \quad (287)$$

PROOF: $E\left[\sum_k \overline{\mathbf{o}}_{jk}\boldsymbol{\Sigma}\,\overline{\mathbf{o}}_{jk}\right]=E\left[\left(\boldsymbol{\mu}-\boldsymbol{\pi}_0\right)'\boldsymbol{\alpha}_{j0}\boldsymbol{\alpha}_{j0}{}'\left(\boldsymbol{\mu}-\boldsymbol{\pi}_0\right)\right]$ reversing steps (266) to (270) (288)

so $E\left[\Pi_j^{new}\right]=-b_jE\left[\left(\boldsymbol{\mu}-\boldsymbol{\pi}_0\right)'\boldsymbol{\alpha}_{j0}\boldsymbol{\alpha}_{j0}{}'\left(\boldsymbol{\mu}-\boldsymbol{\pi}_0\right)\right]$ by (282) (289)

$$=-b_jE\left[\left(\boldsymbol{\mu}-\boldsymbol{\pi}_0\right)'\mathbf{v}_{j0}\mathbf{v}_{j0}{}'\left(\boldsymbol{\mu}-\boldsymbol{\pi}_0\right)\right]-b_jE\left[\left(\boldsymbol{\mu}-\boldsymbol{\pi}_0\right)'\mathbf{u}_0\mathbf{u}_0{}'\left(\boldsymbol{\mu}-\boldsymbol{\pi}_0\right)\right] \quad (290)$$

$$E\left[\Pi_j^{new\,data}\right]=-b_jE\left[\boldsymbol{\mu}'\mathbf{v}_{j0}\mathbf{v}_{j0}{}'\boldsymbol{\mu}\right] \qquad \text{given } \mathbf{v}_{j0}\perp\boldsymbol{\pi}_0 \quad (291)$$

$$=-b_jE\left[\hat{\boldsymbol{\mu}}'\cdot\hat{\boldsymbol{\mu}}\right] \qquad \text{where } \hat{\boldsymbol{\mu}} \text{ is the projection of } \boldsymbol{\mu} \text{ onto } \mathbf{a} \quad (292)$$

$$=-b_j\left|\boldsymbol{\mu}\right|^2E\left[R_j^2\right] \qquad \text{noting } R_j^2=\frac{\text{ESS}}{\text{TSS}}=\frac{\left|\hat{\boldsymbol{\mu}}\right|^2}{\left|\boldsymbol{\mu}\right|^2} \qquad \# \quad (293)$$

RESULT 43. *For new investors not using price:*

$$E\left[\Pi_j^{new}\right]=-b_jE\left[\left(\boldsymbol{\mu}-\boldsymbol{\pi}_0\right)'\mathbf{v}_j\mathbf{v}_j{}'\left(\boldsymbol{\mu}-\boldsymbol{\pi}_0\right)\right] \quad (294)$$

PROOF: As per (288); here $\boldsymbol{\alpha}_{j0}=\mathbf{v}_j$ #

5.4. Total profit

Both *new* and *mature* profits are summarized in the following table. It would enhance the integrity of the model if the elements in the table demonstrably summed to zero but this development has not yet been undertaken.

Investor type		Data profit	Price profit
Using price	*mature*	$+\lvert\boldsymbol{\mu}\rvert\cdot\lvert\boldsymbol{\mu}-\bar{\boldsymbol{\pi}}\rvert\cdot E\left[R_j^2\right]$	$\left(\boldsymbol{\mu}'\cdot E\left[\mathbf{u}_{jt}\mathbf{u}_{jt}'\right]-\bar{\boldsymbol{\pi}}\right)\left(\boldsymbol{\mu}-\bar{\boldsymbol{\pi}}\right)<0$
	new	$-\lvert\boldsymbol{\mu}\rvert\cdot\lvert\boldsymbol{\mu}\rvert\cdot E\left[R_j^2\right]$	$-E\left[\left(\boldsymbol{\mu}-\boldsymbol{\pi}_0\right)'\mathbf{u}_0\mathbf{u}_0'\left(\boldsymbol{\mu}-\boldsymbol{\pi}_0\right)\right]$
Not using price	*mature*	$+\lvert\boldsymbol{\mu}-\bar{\boldsymbol{\pi}}\rvert^2 E\left[R_j^2\right]$	
	new	$-E\left[\left(\boldsymbol{\mu}-\boldsymbol{\pi}_0\right)'\mathbf{v}_j\mathbf{v}_j'\left(\boldsymbol{\mu}-\boldsymbol{\pi}_0\right)\right]$	

TABLE 5: Expected net profit of investor *j* in period 1, classified by investor type and source. Investors may or may not include the price variable in their estimations. The *mature* class of investors formed their estimates sufficiently long ago that current price has no particular relationship to price at the time of estimation. The *new* investors formed their estimates in the most recent period so that the price in the current period is correlated with the price in their estimating period. This affects new investors adversely. The conclusion is that the analysis of non-price data is profitable, and analysis of price data is unprofitable. Entries in the table should be multiplied by the weight of money b_j.

The results cover only *new* investors and *mature* investors, with none for the time intervals between; it is reasonable to suppose that profit expectations for intermediate time periods are intermediate in magnitude, and that the investor's expected profit is some function of the *new* and *mature* expected profits. We assume this is the case:

PREMISE 11: PROFIT INTERPOLATION. Profit can be linearly interpolated from *new* and *mature* profit for data and price.

$$E\left[\Pi_j\right]=c_j^{\substack{mature\\data}}E\left[\Pi_j^{\substack{mature\\data}}\right]+c_j^{\substack{mature\\price}}E\left[\Pi_j^{\substack{mature\\price}}\right]+c_j^{\substack{new\\data}}E\left[\Pi_j^{\substack{new\\data}}\right]+c_j^{\substack{new\\price}}E\left[\Pi_j^{\substack{new\\price}}\right] \qquad (295)$$

$$\text{where}\quad c_j^{\substack{mature\\data}},c_j^{\substack{mature\\price}},c_j^{\substack{new\\data}},c_j^{\substack{new\\price}}>0 \qquad (296)$$

RESULT 44. *Data profit is positive for the average investor.*

$$\sum_j\left(c_j^{\substack{mature\\data}}E\left[\Pi_j^{\substack{mature\\data}}\right]+c_j^{\substack{new\\data}}E\left[\Pi_j^{\substack{new\\data}}\right]\right)>0 \qquad (297)$$

PROOF: $0=\sum_j E\left[\Pi_j\right]$ taking expectations of (230) (298)

$$= \sum_j \left(c_j^{\overset{mature}{data}} E\left[\Pi_j^{\overset{mature}{data}} \right] + c_j^{\overset{new}{data}} E\left[\Pi_j^{\overset{new}{data}} \right] \right) + \sum_j \left(c_j^{\overset{mature}{price}} E\left[\Pi_j^{\overset{mature}{price}} \right] + c_j^{\overset{new}{price}} E\left[\Pi_j^{\overset{new}{price}} \right] \right) \qquad \text{using (295)} \qquad (299)$$

Now $$c_j^{\overset{mature}{price}} E\left[\Pi_j^{\overset{mature}{price}} \right] + c_j^{\overset{new}{price}} E\left[\Pi_j^{\overset{new}{price}} \right] < 0 \quad \text{for all investors } j \qquad (300)$$

so $$\sum_j \left(c_j^{\overset{mature}{price}} E\left[\Pi_j^{\overset{mature}{price}} \right] + c_j^{\overset{new}{price}} E\left[\Pi_j^{\overset{new}{price}} \right] \right) < 0 \quad \text{result follows (299)} \qquad \# \qquad (301)$$

RESULT 45. *If the profit coefficients* $c_j^{\overset{mature}{data}}, c_j^{\overset{mature}{price}}, c_j^{\overset{new}{data}}, c_j^{\overset{new}{price}}$ *are the same for every investor, then for each investor expected profit is a positive function of their coefficient of determination* $E\left[R_j^2\right]$.

$$E\left[\Pi_j\right] = K_j^{data} E\left[R_j^2\right] + K_j^{price} \qquad (302)$$

where $K_j^{data} > 0$ *is independent of the data* $\mathbf{a}_j$ *used by investor j* (303)

and $K_j^{price} = E\left[\Pi_j^{price}\right] < 0$ *is independent of the data* $\mathbf{a}_j$ *used by investor j* (304)

PROOF: $$\sum_j \left(c^{\overset{mature}{data}} b_j |\boldsymbol{\mu}| \cdot |\boldsymbol{\mu} - \overline{\boldsymbol{\pi}}| \cdot E\left[R_j^2\right] - c^{\overset{new}{data}} b_j |\boldsymbol{\mu}| \cdot |\boldsymbol{\mu}| \cdot E\left[R_j^2\right] \right) > 0 \quad \text{Table 5, (297)} \qquad (305)$$

$$\left(c^{\overset{mature}{data}} |\boldsymbol{\mu}| \cdot |\boldsymbol{\mu} - \overline{\boldsymbol{\pi}}| - c^{\overset{new}{data}} |\boldsymbol{\mu}| \cdot |\boldsymbol{\mu}| \right) \sum_j b_j E\left[R_j^2\right] > 0 \qquad \text{rearranging} \qquad (306)$$

$$c^{\overset{mature}{data}} |\boldsymbol{\mu}| \cdot |\boldsymbol{\mu} - \overline{\boldsymbol{\pi}}| - c^{\overset{new}{data}} |\boldsymbol{\mu}| \cdot |\boldsymbol{\mu}| > 0 \qquad \text{given } \sum_j b_j E\left[R_j^2\right] > 0 \qquad (307)$$

so applying this to the expected data profit of a single investor,

$$E\left[\Pi_j^{data}\right] = \left(c^{\overset{mature}{data}} |\boldsymbol{\mu}| \cdot |\boldsymbol{\mu} - \overline{\boldsymbol{\pi}}| - c^{\overset{new}{data}} |\boldsymbol{\mu}| \cdot |\boldsymbol{\mu}| \right) b_j E\left[R_j^2\right] \qquad (308)$$

$$= K_j^{data} R_j^2 \qquad (309)$$

where $$K_j^{data} = \left(c^{\overset{mature}{data}} |\boldsymbol{\mu}| \cdot |\boldsymbol{\mu} - \overline{\boldsymbol{\pi}}| - c^{\overset{new}{data}} |\boldsymbol{\mu}| \cdot |\boldsymbol{\mu}| \right) b_j > 0 \qquad \text{by (13),(307)} \qquad (310)$$

The expected price profit

$$E\left[\Pi_j^{price}\right] = -\left(|\boldsymbol{\mu}| \cdot |\boldsymbol{\mu} - \overline{\boldsymbol{\pi}}| \cdot \frac{\lambda_X}{\lambda_P} \operatorname{var} x \right) - E\left[\left(\boldsymbol{\mu} - \boldsymbol{\pi}_0\right)' \mathbf{u}_0 \mathbf{u}_0' \left(\boldsymbol{\mu} - \boldsymbol{\pi}_0\right) \right] < 0 \qquad (311)$$

is not a function of data variables. #

The premise of equal profit coefficients is one way of developing this result. Another way is to interpret the previous result as an expectation over investors, but this does not guarantee that $K_j^{data} > 0$ for every investor.

5.5. Long term expectations and rational learning

We now ask whether the behavioural premises 7 and 8 are consistent with 'rational learning' as defined by premise 1. We judge realizations by the expected profit given in Table 6 above and this is to be compared with the predicted returns (20) obtained from estimation. We see that the return to *new* investors is less than the return to mature investors which is perhaps surprising given that their estimates are more up to date. The short term profit $\Pi^{no\ pricemove}$ which would be earned in estimation period 1 in the absence of a price move is positive as expected, but it turns out that the situation is not as simple as this and we have to consider price shift and estimation error, both of which affect the investor adversely. As the offset $\boldsymbol{\mu} - \bar{\boldsymbol{\pi}}$ is small, we expect that for new investors the short run loss will exceed the long run profit components, and overall profit will be negative. As the price shift and misestimation factors wear off over time as the price and estimation matrix move, profit will increase and eventually become positive. This progression is consistent with an overall profit across all investors of zero. The profit outcomes, and hence long term expectations and behaviour, of the investor are compatible with the assumed behaviour of the investor insofar as:

- *Estimating*: The investor must start off by estimating notwithstanding the initial loss. Furthermore, the long term profit value given above is an expected value, not a definite amount, and the investor must sample several times to be sure of having a sufficiently representative sample. There is also the matter of structural change in the return relationship (10).
- *Not estimating*: As positive profit comes only over time, the investor will not reestimate in every period to get fresh estimates, so the majority of investors will be *old* investors as assumed.
- *Investor demand function*: The return which is received will ultimately be positive so the investor is justified in using the relationship (11) to determine the quantity which they invest. However, the investor will find that realized return is less than the predicted return $\hat{r}_j$. This will be reflected in somewhat smaller values of the weight of money B_j and the proportion of *new* investors dw than would be the case if expectations were exactly realized.

As stability condition (215) requires a relatively low proportion of *new* investors, it appears that the failure of investors to fully realize the return which the regressions indicate is important for the stability of the market.

5.6. The viability of non-price data

A primary concern of the literature dealing with the Grossman Stiglitz paradox has been that low-cost price watchers can outperform high cost fundamental analysts, reducing the viability of fundamental analysis and disrupting market efficiency. Consider that some data costs virtually nothing to compile yet can have considerable explanatory value: the constant vector ι and time period dummy variables.

DEFINITION: RELEVANT VARIABLE. A variable is relevant if it is correlated with $\boldsymbol{\mu} - \bar{\boldsymbol{\pi}}$.

DEFINITION: DISTINCT VARIABLE. A relevant variable is considered distinct if it cannot be expressed as a linear sum of other relevant variables.

PREMISE 12: INVESTORS' GOAL. Investors will maximize expected net profit Π^{net} which is a function of the gross profit given in (302) and the cost of the data.

$$E\left[\Pi_j^{net}\right] = K_j^{data} E\left[R_j^2\right] + K_j^{price} - C\left(\mathbf{a}_j\right) \tag{312}$$

where $C\left(\mathbf{a}_j\right) = \sum_k C\left(\mathbf{a}_{jk}\right)$ (313)

is the total cost of data which is an arithmetic sum of the cost of each data series $\mathbf{a}_{jk}$ used by the investor.

PREMISE 13: DATA CHARACTERISTICS. At least two relevant, distinct and free data series are available.

RESULT 46: VIABLE MARKET THEOREM. *Investors will use data other than price.*

PROOF: $E\left[\Pi_j^{marginal}\right] = E\left[\Pi_j^{net} \middle| 2 \text{ data+price}\right] - E\left[\Pi_j^{net} \middle| \text{price alone}\right]$ (314)

$$= K_j^{data}\left(E\left[R_j^2 \middle| 2 \text{ data+price}\right] - E\left[R_j^2 \middle| \text{price alone}\right]\right) - C\left(\mathbf{a}_{j2}\right) - C\left(\mathbf{a}_{j3}\right) \tag{315}$$

noting that K_j^{data} and K_j^{price} are independent of data $\mathbf{a}_j$ used by the investor

$$= K_j^{data}\left(E\left[R_j^2 \middle| 2 \text{ data+price}\right] - E\left[R_j^2 \middle| \text{price alone}\right]\right) \quad \text{as } C\left(\mathbf{a}_{j2}\right) + C\left(\mathbf{a}_{j3}\right) = 0 \tag{316}$$

Now $E\left[R_j^2 \middle| 2 \text{ data+price}\right] > E\left[R_j^2 \middle| \text{price alone}\right]$ (317)

as price cannot be collinear with two distinct variables, by Premise 13 they are relevant.

so $E\left[\Pi_j^{marginal}\right] > 0$ as both factors in (316) are positive. #

If it is accepted that a certain amount of free and useful data is available then this result demonstrates that a data set is always available in the face of strategy switching. Premise 3 of the objective model is upheld (i.e. non-price data is used by investors) and market processes are viable in the long term. While it has not been demonstrated formally that the system will find its way to a sustainable equilibrium from any starting point, it seems clear intuitively. If less data is being used, the investor will add the free series. If more data is being used and cost makes the investor's position untenable, the investor will drop back to the free set. In fact more than two sources of free data are needed to avoid instability due to relatively large collinear variance: the criterion identified at (223).

5.7. The viability of price data

We turn to the problem, which is probably unique in the history of literature inspired by Grossman Stiglitz (1980), of having to explain why investors would be motivated to look at price. The importance of price in the investment decision would seem to be self-evident; but the preceding analysis demonstrates that the imprecise nature of the price variable – which wanders through coefficient price – makes it a poor guide to where positive returns might be found. In the short term estimation error destroys its value and in the long term it is too vague. In both cases the losses of the price-watchers subsidize the investors using 'hard' data.

In terms of the objective model as set out here the answer is as follows: if some investors stop looking at price then the proportion of those who do, b^{price}, will fall. The stability condition:

$$dw < \frac{\sigma_X^2}{2}\left(\frac{b_{price}}{\lambda_P}\right)\left(\frac{\lambda_X}{\lambda_P}\right)^2 \qquad \text{(215) restated} \qquad (318)$$

implies that past a certain point in this fall in b^{price} the market will become unstable and bubble. It is reasonable to suppose (although it is not proved here) that in a bubble the investors who consider price will ultimately profit and those who do not will incur losses. There is a long run cycle within which price watching behaviour is rewarded.

6. Heterogeneous least squares learning as a genetic algorithm

Any process which uses a blind multipronged hill-climbing algorithm can be described as a genetic algorithm because this is the presumed mechanism of natural selection. We show that heterogeneous least squares learning is a genetic algorithm by constructing a genetic algorithm for a natural system and showing that it operates identically to heterogeneous least squares learning within the objective model.

The similarity of the processes does not depend on the profitability and survival of the economic actors, which is the traditional avenue for constructing an analogy of the 'economic Darwinism' type. Nor does it depend on the selection of particular strategies, which is the version of this concept found in modern rational expectations literature. For instance Marimon McGrattan (1995) demonstrate an isomorphism between adaptive learning and evolutionary learning. The process described by the objective model will operate without any elimination of less profitable investors or strategies. It is the information processing itself – least squares learning – which is analogous to natural selection.

Let a creature *j*, i.e. a single organism, be represented by an *N**1 vector $\mathbf{g}_j$ in 'gene space'. Each gene represents a separate dimension of variation, variations are ranked in some physical order, and there are *N* separate genes so gene space is $\mathbb{R}^N$. It is a question whether genetic variation has a continuous character but we will ignore this. Let the population mean of the **g** vectors be denoted by $\boldsymbol{\pi}$, and the optimal genotype - the genotype which is superior to all others in reproductive capability - by $\boldsymbol{\mu}+\mathbf{e}$. Optimum fitness has a stochastic component, denoted **e**, which is set by prevailing environment conditions, the situation with predator/prey species etc. We suppose that the mean is sufficiently close to the optimum that reproductive capability is a smooth concave function of $\boldsymbol{\pi}$**,** with maximum at $\boldsymbol{\mu}+\mathbf{e}$. For the species this presents as a hill climbing problem.

Let the relative weight of creature *j* (the number of individuals with genotype $\mathbf{g}_j$) be denoted by b_j. At period 0

$$1 = \sum_j b_j \tag{319}$$

$$\boldsymbol{\pi}_0 = \sum_j b_j \mathbf{g}_j \tag{320}$$

Variation around the mean, referred to as the 'variation' vector, can be denoted by $\boldsymbol{\alpha}_j$:

$$\boldsymbol{\alpha}_j = \mathbf{g}_j - \boldsymbol{\pi}_0 \tag{321}$$

Observe $\sum_j b_j \boldsymbol{\alpha}_j = \mathbf{0}$ (322)

The population fitness vector $\mathbf{r}$ is the vector from the population mean $\boldsymbol{\pi}$ to the current optimum $\boldsymbol{\mu} + \mathbf{e}$.

$$\mathbf{r}_0 = \boldsymbol{\mu} + \mathbf{e}_0 - \boldsymbol{\pi}_0 \tag{323}$$

NATURAL SELECTION PREMISE 1. The fitness of a each creature is given by the distance $\left|\boldsymbol{\mu} + \mathbf{e}_0 - \mathbf{g}_j\right|$ of its genotype $\mathbf{g}_j$ from the optimum genotype $\boldsymbol{\mu} + \mathbf{e}_0$. The fitness of the mean genotype is given by the distance $\left|\boldsymbol{\mu} + \mathbf{e}_0 - \boldsymbol{\pi}_0\right|$. The relative fitness of a creature is the difference between its fitness and the population fitness and this is approximately proportional to the coordinate of its fitness vector in the direction of the population fitness vector.

$$relative\ fitness_j = \left|\boldsymbol{\mu} + \mathbf{e}_0 - \mathbf{g}_j\right| - \left|\boldsymbol{\mu} + \mathbf{e}_0 - \boldsymbol{\pi}_0\right| \tag{324}$$

$$\propto \boldsymbol{\alpha}_j'\mathbf{r} \tag{325}$$

NATURAL SELECTION PREMISE 2. The Net Reproduction Rate (*NRR*) of a creature (i.e. production of offspring after replacing itself) is proportional to the relative fitness of the creature.

$$NRR_j = w \cdot relative\ fitness = w\boldsymbol{\alpha}_j'\mathbf{r} \tag{326}$$

where *w* is some positive constant. *NRR* will be positive for some creatures and negative for others; on average it is zero as shown in (329). The Gross Reproduction Rate (*GRR*), which includes the replacement creature, is given by

$$GRR_j = NRR_j + 1 = w\boldsymbol{\alpha}_j'\mathbf{r} + 1 \tag{327}$$

so the weighting of descendents produced by creatures of type *j* will be given by

$$b_{j\ descendents} = b_j \cdot GRR_j = b_j\left(1 + w\boldsymbol{\alpha}_j'\mathbf{r}\right) \tag{328}$$

Observe that the total weighting attributed to descendents is unity:

$$\sum_j b_{j\ descendents} = \sum_j b_j\left(1 + w\boldsymbol{\alpha}_j'\mathbf{r}\right) = \sum_j b_j + w\left(\sum_j b_j \boldsymbol{\alpha}_j'\right)\mathbf{r} = 1 \quad \text{by (319),(322)} \tag{329}$$

so these weights $b_{j\ descendents}$ can be used for the purpose of finding the new population mean in period 1.

NATURAL SELECTION PREMISE 3. The average genotype $\overline{\mathbf{g}_{ij}}$ of the descendents *i* of creature *j* equals the genotype of the parent, i.e.

$$\mathbf{g}_{ij} = \mathbf{g}_j + \mathbf{u}_i \qquad \text{where } \mathbf{u}_i \text{ is an error term} \tag{330}$$

$$\overline{\mathbf{g}_{ij}} = \frac{\sum_i \mathbf{g}_{ij}}{GRR_j} = \mathbf{g}_j \tag{331}$$

RESULT 47: GENETIC ALGORITHM THEOREM. *Heterogeneous least squares learning is a genetic algorithm.*
PROOF: The above three assumptions define a genetic algorithm. We proceed by showing that this algorithm embodies the same process as heterogeneous least squares learning.
Evaluate the population mean in period 1, $\boldsymbol{\pi}_1$.

$$\boldsymbol{\pi}_1 = \sum_j \overline{\mathbf{g}_{ij}} \cdot b_{j\ descendents} \tag{332}$$

$$= \sum_j \left(\boldsymbol{\pi}_0 + \boldsymbol{\alpha}_j\right) \cdot b_j \left(1 + w\boldsymbol{\alpha}_j{}'\mathbf{r}\right) \qquad \text{by (331),(321),(328)} \tag{333}$$

$$= \sum_j b_j \boldsymbol{\pi}_0 + \sum_j \boldsymbol{\pi}_0 b_j w \boldsymbol{\alpha}_j{}'\mathbf{r} + \sum_j \boldsymbol{\alpha}_j b_j + \sum_j \boldsymbol{\alpha}_j b_j w \boldsymbol{\alpha}_j{}'\mathbf{r} \tag{334}$$

$$= \sum_j b_j \boldsymbol{\pi}_0 + \mathbf{0} + \mathbf{0} + w\sum_j b_j \boldsymbol{\alpha}_j \boldsymbol{\alpha}_j{}'\mathbf{r} \qquad \text{by (322)} \tag{335}$$

$$= \boldsymbol{\pi}_0 + w\mathbf{H}\left(\boldsymbol{\mu} - \boldsymbol{\pi}_0 + \mathbf{e}_0\right) \qquad \text{using (323)} \tag{336}$$

where $\mathbf{H} = \sum_j b_j \boldsymbol{\alpha}_j \boldsymbol{\alpha}_j{}'$ (337)

Effectively each creature is forming a regression coefficient $\boldsymbol{\alpha}_j{}'\left(\boldsymbol{\mu} - \boldsymbol{\pi}_0\right)$ which is applied to explanatory variable $\boldsymbol{\alpha}_j$ to yield prediction vector $\boldsymbol{\alpha}_j \boldsymbol{\alpha}_j{}'\left(\boldsymbol{\mu} - \boldsymbol{\pi}_0\right)$. Matrix **H**, the weighted sum of the projection matrices $\boldsymbol{\alpha}_j \boldsymbol{\alpha}_j{}'$, is cognate with the estimation matrix **H** defined at (85).
Now this expression can be rearranged and expectations taken to give

$$E\left[\boldsymbol{\mu} - \boldsymbol{\pi}_1\right] = E\left[\boldsymbol{\mu} - \boldsymbol{\pi}_0\right] - w\mathbf{H} \cdot E\left[\boldsymbol{\mu} - \boldsymbol{\pi}_0\right] \tag{338}$$

$$= \left(\mathbf{I} - w\mathbf{H}\right) \cdot E\left[\boldsymbol{\mu} - \boldsymbol{\pi}_0\right] \qquad \text{which conforms with (131)} \qquad \# \tag{339}$$

One apparent difference is that the vectors $\boldsymbol{\alpha}_j$ are normalized in the financial model, i.e. $\boldsymbol{\alpha}_j{}'\boldsymbol{\alpha}_j = \mathbf{I}$ whereas this is not the case for the genetic algorithm; but the scaling of genetic variation is in any case arbitrary. The estimation matrix **H** is a genetic operator – it takes the system state (price, mean genotype) and an observation of the environment (return, mean fitness) and produces the next state.

The following figures and table examine the correspondence between financial and natural systems for its own interest.

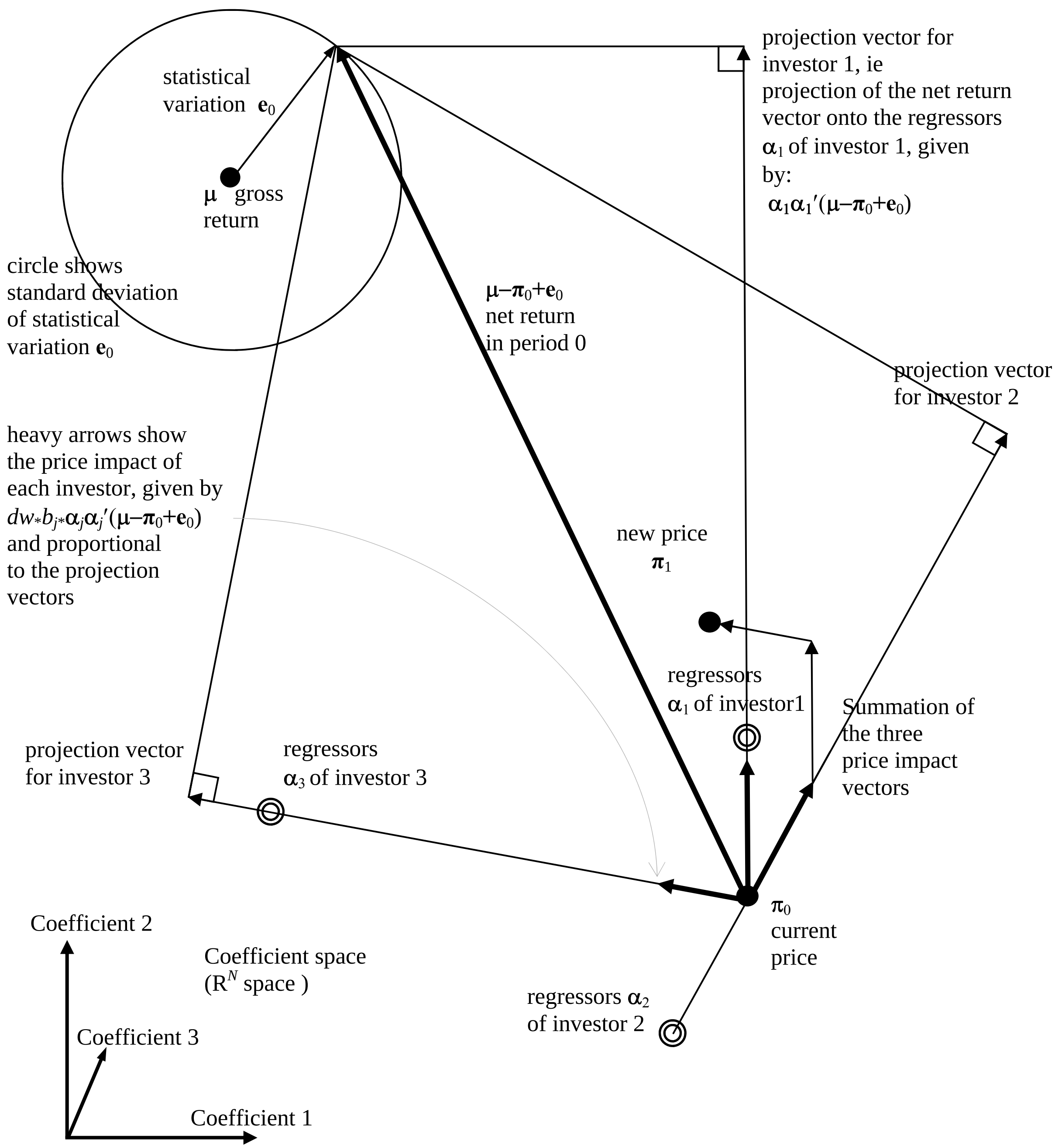


FIG. 5. **Objective model applied to finance**. The market is initially at price $\boldsymbol{\pi}_0$, the lowest ● symbol. Each investor attempts to find a profitable strategy by regressing the net return $\boldsymbol{\mu} + \mathbf{e}_0 - \boldsymbol{\pi}_0$ onto particular regressor vectors $\boldsymbol{\alpha}$ which they have chosen, shown by the target symbols ◎. This yields a predicted return given by the geometric projection of the net return vector onto the regressors. The consequent impact of the investor's position on the price is in the direction of the regressors $\boldsymbol{\alpha}$ and proportional to this geometric projection: the impact is shown by the heavy arrows. Effect is to move the price to $\boldsymbol{\pi}_1$ in the next generation and ultimately to the gross return vector $\boldsymbol{\mu}$.

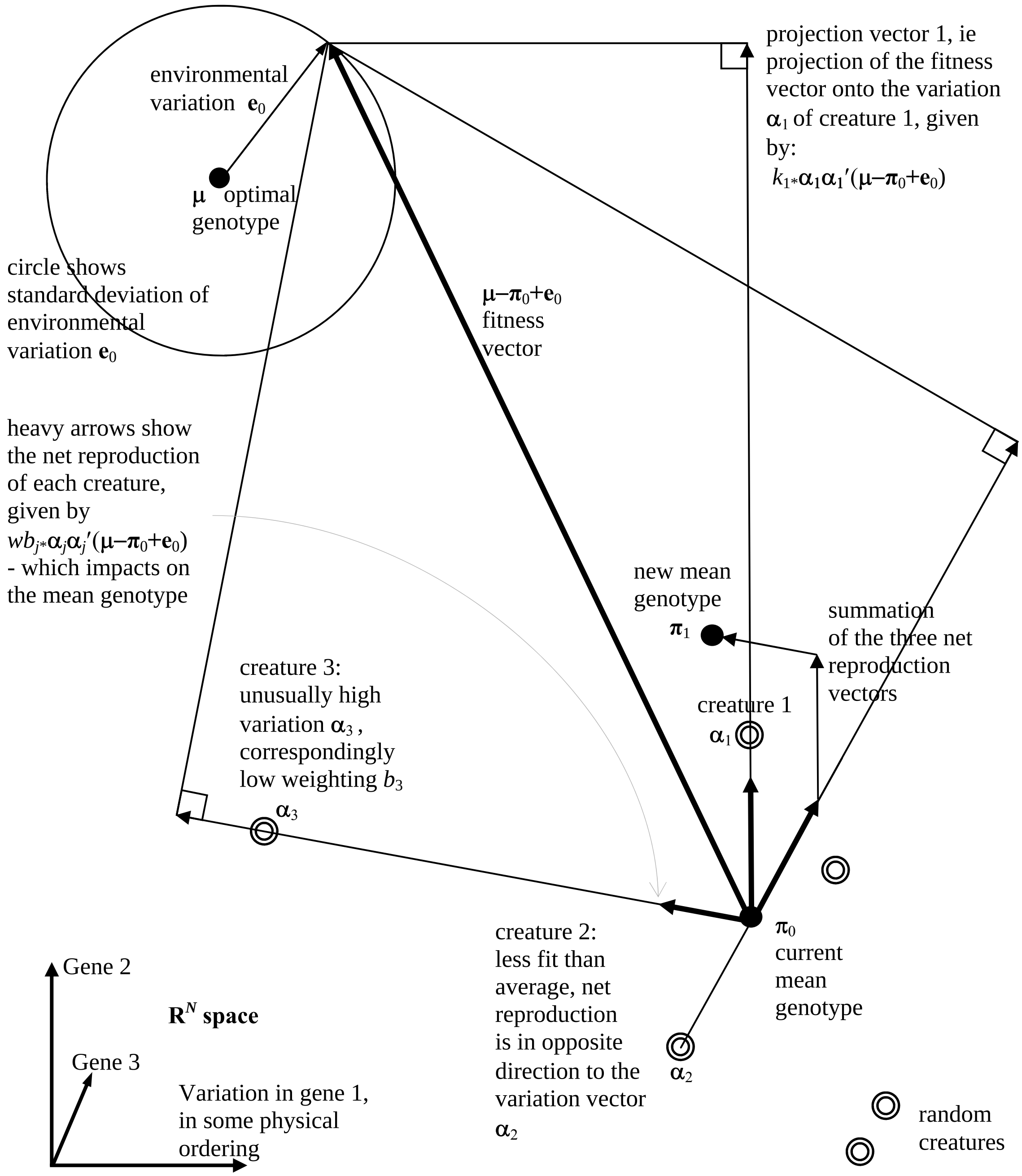


FIG. 6. **Objective model applied to natural selection**. The species initially has mean genotype $\boldsymbol{\pi}_0$, the lowest ● symbol, and individual creatures are shown by the target symbols ◎. Each creature's fitness is determined by the covariance of its genetic variation $\boldsymbol{\alpha}$ and the fitness vector $\boldsymbol{\mu}+\mathbf{e}_0-\boldsymbol{\pi}_0$, suggested by geometric projection of the fitness vector onto the variation line $\boldsymbol{\alpha}$. Consequent net reproduction of the creature perpetuates its variation $\boldsymbol{\alpha}$, but is proportional to fitness: net reproduction is shown by the heavy arrows. Effect is to move the mean genotype to $\boldsymbol{\pi}_1$ in the next generation and ultimately to optimal fitness $\boldsymbol{\mu}$.

TABLE 7: Correspondence of the elements in financial and natural systems.

	FINANCE	NATURAL SELECTION
STARTING POINT	*Price* Price is a vector $\boldsymbol{\pi}_0$ in 'coefficient space', not a single piece of information; it stores all the information which generates an accurate price.	*Mean genotype* The mean genotype is a vector $\boldsymbol{\pi}_0$ in 'gene space'**.** A creature is a function, not a single piece of information, formed by all the information in its gene vector (set of chromosomes).
CREATION OF VARIATION	Different variables are selected by actors for investigation. Each actor *j* relies on the price to explain the majority of return, and chooses only a small number of variables $\boldsymbol{\alpha}_j$ for further investigation in a process of econometric *specification*.	Different genotypes are created by genetic crossover, i.e. for sexually reproducing species this comprises meiosis & fertilization. Different creatures *j* show variation along every genetic axis to produce genotype $\mathbf{g}_j$ with variation $\boldsymbol{\alpha}_j$ from the mean.
TEST OF VARIATION	Set of statistical *trials i* forming an *estimation period*, in which security returns and explanatory variable values are generated.	The creature's efforts to reproduce represent separate statistical *trials*.
EVALUATION OF TEST	*Estimation* i.e. regression to determine new estimates.	*Natural selection*, i.e. differential rates of survival. As shown above, this is equivalent to an estimation by regression.
TEST ERROR	The accuracy of estimates is affected by error (statistical variation), denoted **e** in coefficient space.	Creature lives are affected by random variations – different environmental conditions, different numbers of predators and prey at various times etc. (The role of chance has received attention in ecological literature in recent years.)
RECOMBINATION	Intermixing of the new estimates and the old through a process of price formation to generate a new *price* $\boldsymbol{\pi}_1$.	Intermixing of members of the next generation of the population to produce a new *mean genotype* $\boldsymbol{\pi}_1$.

7. Conclusion

7.1. Economics of the efficient market and the Grossman-Stiglitz paradox

Financial markets are a challenge to rational expectations theorists because they are a zero sum game. For expectations to be 'rational' requires that they be realized to some extent, yet it is not possible for everybody to have a positive expected return in the single period model. This contradiction is an alternative way of stating the Grossman Stiglitz paradox which puts the emphasis on expectations rather than behaviour.

The properties exhibited by the market in the objective model suggest that efficiency is a journey rather than a destination. It is the expected value of price which approximately equals return: $E\left[\boldsymbol{\pi}\right] \approx \boldsymbol{\mu}$ rather than price itself which equals return: $\boldsymbol{\pi} = \boldsymbol{\mu}$. The process by which price gravitates to return is a stochastic one in which investors are rewarded according to their contributions. As in the original Grossman Stiglitz (1980) model it is noise which guarantees the economic viability of the process, but unlike Grossman Stiglitz and like Goldbaum (2005, 2006) the noise is generated endogenously by the learning process.

Specifically, the return received by each investor is determined by:

- use (or not) of non-price data, for which expected return is positive in a stable market
- the relevance R^2 of the investor's non-price information
- use (or not) of price data, for which expected return is negative in a stable market and presumed to be positive in a bubble/crash market
- the sampling error in the estimate derived from the information
- the timeliness of the estimates (*new*, *old* or *mature*)
- whether the market is in stable or bubble mode

and the returns may be positive or negative depending on these factors. All of these factors could be predicted on a priori grounds, and most are related to profit as one would expect intuitively – except that *new* estimates are expected to lose money because of statistical error, and price estimates do not yield a positive return in a stable market. Many of the models in the literature are driven on cost and strategy switching assumptions. Within the context of these models, too high a level of cost causes investors to switch into price watching behaviour, which causes the market to falter. Within the objective model it is always possible for the market to make a price even when fundamental analysis is costly and investors switch away from it, because of the existence of useful but free regressors such as the constant regressor ι_{T*1} and time related dummy variables.

The objective model suggests that when the market is stable, the return to the price variable will be negative because the price vector moves in coefficient space and the price coefficient does not estimate well. Investors who use price as a variable will lose money on that account (although they may make money on other variables) and subsidize the other market participants. The negative return will discourage the use of the price variable in stable markets. In bubble/crash conditions it is assumed (although it has not been shown) that use of the price variable will be profitable because the investor will avoid the crash and the ranks of the price watchers will be replenished. This suggests that rather than being an aberration, price bubbles are intrinsic to the way that markets work. The market cycles between stable and bubble modes as the proportion of investors using price, b_{price}, falls in stable times and rises after market crashes.

The disposition of returns in the objective model is consistent with sustainable patterns of behaviour and this points the way to the resolution of the Grossman Stiglitz paradox. It appears that a rational expectations model is capable in principle of explaining the observed facts of financial markets and that recourse to ad hoc or behavioural assumptions is not necessary.

7.2. The fragile nature of financial market processes

The process of forming a price in financial markets is inherently fragile because it relies on the generation of a negative price coefficient to act as denominator, and it turns out that the price coefficient is close to zero. Amplifying the denominator problem is the instability which probably attends any system where expectations of positive return have to be reconciled with a net realization of zero. It is only a fortuitous interplay of parameters which allows markets to operate at all, and it is not surprising that markets are susceptible to occasional malfunction for a variety of different reasons.

Three specific mechanisms for market bubbles have been identified. They arise as natural consequences of objective theory rather than a conscious attempt to model bubbles. Market efficiency and market bubbles can be regarded as two sides of the same coin: if the market is not heading to the efficient point it is heading to infinity.

- The stability condition of the price convergence process is not satisfied, because the frequency with which investors update their estimates, dw, is too great.

  $$dw > \frac{\sigma_X^2}{2}\left(\frac{b_{price}}{\lambda_P}\right)\left(\frac{\lambda_X}{\lambda_P}\right)^2 \quad \text{(215) restated with sign reversed} \qquad (340)$$

- There is insufficient variance in the component of return which is orthogonal to the true value of return to generate a negative value for the price coefficient. In practice this means that investors are only looking at a small range of information. A rule of thumb derived from simulation testing is that instability sets in where:

  $$r_{error} = \frac{\sigma_{error\ y}}{\sigma_{error\ x}} > 20\% \quad \text{(221) restated} \qquad (341)$$

- The analysis of profit demonstrates that the returns to price watching are negative in the short term. This may lead investors to ignore price and concentrate on other variables so that the proportion of investors using price b_{price} falls. As a result stability condition (340) is not satisfied. It is conjectured that the consequent bubble/crash will reward price watching and lift b_{price}.

The practical effect of each of these mechanisms is that the price coefficient is too close to zero for the learning process to operate; the stability condition is not satisfied and deviations of price from return are magnified rather than damped. In practice these situations are particularly likely for startup issues which are not yet making a profit in a time of buoyant economic growth. There is too little non-price information available and investors are concentrating on price alone. Three famous historical bubbles fall into this category: the Dutch tulip mania of 1636-37, the South Seas bubble of 1720, and Dot-com mania in 2000. It is common to suppose that bubbles occur despite the fact that investors realize that stocks are overpriced, because each investor thinks they can find a 'greater fool' to sell to. The mechanisms given here flesh out that intuition in the context of learning models.

7.3. The coefficient superstrate and the reinterpretation of price as an object

When the original data model is reduced to coefficient space every variable and process takes a new form. The representation of the least squares learning process in coefficient space is simpler and the behaviour of the market is more intuitive. Coefficient space can be understood as a hidden superstrate which constructs the market. This expression 'constructs the market' can be given a formal meaning:

In any well-formed explanation of why the market price tends towards the realized return in a heterogeneous least squares learning market, the data $\mathbf{X}$ *will drop out and we will be left with the coefficient representation.*

Although the word 'substrate' is used in this paper's abstract, the less familiar term 'superstrate' is more accurate in describing the concept of the coefficient layer. 'Constructs the market' is used rather than 'determines market behaviour' because from a causal point of view it is the investors operating at the data level who determine market behaviour. Notwithstanding, this behaviour can only be understood at the superstrate level.

We have noted that price is not a single value which is immediately dependent on demand and supply conditions but a vector with a persistent value. Viewed from within the superstrate, price can be regarded as an object in the sense that this word is used in computing; viz. it is an independent entity with its own properties and methods. Its property is a collection $\boldsymbol{\pi}$ of information. Its methods are those of a computer memory: it stores the information, makes it available to investors via estimation (34) and updates according to the price change theorem (62). The process by which price gravitates to the return can be regarded as emergent behaviour – it is not operating on the data level which is seen by the participants nor do they need to be aware of it.

It follows from the objective existence of price that price fluctuations will occur as the price vector $\boldsymbol{\pi}$ rotates around return $\boldsymbol{\mu}$. These endogenous gyrations suggest the market will exhibit mean reversion and may provide a theoretical basis for that army of practitioners who claim to discern such patterns through ‘technical analysis’. The implications of technical analysis are beyond the scope of this paper which is concerned only with fundamental analysis.

7.4. The investor’s life in the bush of ghosts

One of the oldest and most important themes in economics, going back to Adam Smith’s “invisible hand”, is that the whole is greater than the sum of the parts. The objective model locates the parts which have previously been obscured. Behind the outward appearance of a market – the trades and the current market price – is a hidden superstrate, in which price is not a single piece of information but an object with its own independent existence as an economic entity. The object stores information and makes it available to traders in a same way as a set of genes in biology. Heterogeneous least squares learning is a genetic operator which moves price to the point of optimum explanatory power. The gravitation of price to the efficient point is emergent behaviour in that it cannot be determined from the motivations or initial information of the investors. The process is as invisible to the investors as natural selection is to creatures.

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