Infinite Horizon Hydroelectrictiy Games

Abstract

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1 Introduction

My aim in this paper is to compare infinite and finite horizon models of the interaction between a hydro power generator and a thermal power generator. I use Crampes and Moreaux (2001) (CM henceforth) as the basis for my comparison.

This paper is **preliminary and incomplete.** Comments are very welcome. My apologies to all the authors who deserve citation, but as yet have received none.

2 Model

I model here a hydro power producing dam in Cournot competition with a thermal producer of electricity. We do not presume any commitment power on the part of the Hydro plant, and ask if the dynamic nature of his problem, as compared to the static nature of Thermal plant, is going to result in an endogenous commitment power. For our analysis to have relevance to the problems of power generation, we need to take account of the fact that power demand fluctuates through time. It varies on a monthly basis in response to heating and cooling needs. It also fluctuates through the day in response to, for example, sleep patterns. We do presume that one year is identical to the next. However, within each year, we allow $T \geq 1$ heterogeneous periods. Hence, in our analysis period t has the same exogenous details as period $t + T$. Let $n(t)$ denote the remainder of t/T. That is, if there is some integer x such that $t = xT + j$, then

$n(t) = j$. (As written, we have $n(t) \in \{0, ..., T-1\}$.)

In period t, Thermal generates q_t units of electricity, and Hydro generates h_t units of electricity. In period t, the inverse demand for electricity is $P_t = a_t - b_t(q_t + h_t)$. Given our assumption of a yearly cycle, $a_t = a_{n(t)}$ and $b_t = b_{n(t)}$. Thermal has marginal costs of $mc = c + z \cdot q_t$. Let ψ_t and U_t denote Thermal's instantaneous profits and value function for time t. Let r denote the discount rate. Thermal's objective is

$$\max_{q_t} \psi_t(h_t, q_t) + rU_{t+1}(R_{t+1}) \tag{1}$$

We place no non-negativity constraint on Thermal. This would probably not be worth noting, except that we do place one on Hydro. However, the dynamic nature of Hydro's problem means that interesting effects can arise from his non-negativity constraint. However, nothing out of the ordinary would arise from including a non-negativity constraint for Thermal. Because Thermal has no control of the state variable, his incentives are captured with the first order condition in q_t;

$$\frac{\partial \psi_t}{\partial q_t} = 0 \tag{2}$$

Since ψ_t is not a function of R_t, Thermal's decisions are static in nature. In fact, the only sense in which the choice of q_t is dynamic is that the first order condition depends upon h_t which is chosen dynamically. Let $Q_t(R_t, h_t)$ denote the dependence of Thermal's choice as a function of the state and Hydro's output. From the arguments just made, it follows that $\frac{\partial Q_t}{\partial R_t} = 0$.

Hydro is assumed to have no marginal costs, but to face a resource constraint on water. In particular, Hydro's reserve of water evolves according to $R_{t+1} = R_t + w_t - h_t$, where w_t is the inflow of water. The yearly cycle requires that $w_t = w_{n(t)}$. In period t, Hydro faces three constraints. The *current capacity constraint* requires that he not use more water than currently available, so he must leave $R_{t+1} \geq 0$ (multiplier λ_t.) Let $\bar{R}$ denote the maximum capacity of Hydro's reservoir. The *overflow constraint* requires that Hydro not allow his reservoir to overflow, which requires that $R_{t+1} \leq \bar{R}$ (multiplier θ_t.) We assume throughout that $w_t < \bar{R}$. Hence, the overflow constraint cannot bind if we enter the current period with no water. Finally, Hydro can't produce negative energy, $h_t \geq 0$ (multiplier δ.)

Let π_t and V_t denote Hydro's instantaneous profits and value function for time t. Hydro's objective is

$$\max_{h_t} \pi_t(h_t, q_t) + rV_{t+1}(R_{t+1}) \tag{3}$$

$$\begin{aligned} \text{subject to} \quad R_{t+1} &= R_t + w_t - h_t && (4) \\ R_{t+1} &\geq 0 \text{ with multiplier } \lambda_t && (5) \\ R_{t+1} &\leq \bar{R} \text{ with multiplier } \theta_t && (6) \\ h_t &\geq 0 \text{ with multiplier } \delta_t && (7) \end{aligned}$$

Note that we need the t subscript on π_t and V_t because different periods have different values for a_t (demand parameter) and w_t (water inflows.) Hydro has a first order condition in h_t of

$$r\frac{dV_{t+1}}{dR_{t+1}} = \frac{\partial \pi_t}{\partial h_t} - \lambda_t + \theta_t + \delta_t \tag{8}$$

let $H_t(R_t, q_t)$ denote the dependence of Hydro's choice on the state and Thermal's output. Because H_t and Q_t solve the maximization problems,

$$V_t(R_t) = \pi_t(H_t, Q_t) + \lambda_t(R_t + w_t - H_t) + \theta_t(\bar{R} - R_t - w_t + H_t) + \delta_t h_t + rV_{t+1}(R_t + w_t - H_t)$$

This equation has an envelope condition in R_t of

$$\frac{dV_t}{dR_t} = \frac{\partial \pi_t}{\partial h_t} \cdot \frac{dH_t}{dR_t} + \frac{\partial \pi_t}{\partial q_t} \cdot \frac{dQ_t}{dR_t} + \left(\lambda_t - \theta_t + r\frac{dV_{t+1}}{dR_{t+1}}\right)\left(1 - \frac{dH_t}{dR_t}\right) + \delta_t \cdot \frac{dH_t}{dR_t} \tag{9}$$

Hydro's Euler equation arises from combining Equations 8 and 9. Before writing out the Euler equation, we make it more precise with the following observation. Because $\frac{\partial Q_t}{\partial R_t} = 0$, it follows that $\frac{dQ_t}{dR_t} = \frac{\partial Q_t}{\partial h_t} \cdot \frac{\partial h_t}{\partial R_t}$. It now follows that Hydro's Euler equation is

$$\frac{\partial \pi_t}{\partial h_t} - \lambda_t + \theta_t + \delta_t = r\left(\frac{\partial \pi_{t+1}}{\partial h_{t+1}} + \frac{\partial \pi_{t+1}}{\partial q_{t+1}} \cdot \frac{\partial Q_{t+1}}{\partial h_{t+1}} \cdot \frac{\partial H_{t+1}}{\partial R_{t+1}} + \delta_{t+1}\right) \tag{10}$$

Because we assume zero variable costs of energy production for the Hydro Plant, the derivative on the Left Hand Side of Equation 10 is marginal revenue in period t. The

Right Hand Side of Equation 10 is the shadow price of water in period t.

Rather than work with our specific functional form, we attempt a characterization which requires only that $\frac{\partial Q_{t+1}}{\partial h_{t+1}} > 0$. We break the analysis of Equation 10 into a number of different circumstance. Before doing so, we consider the relationship between the three constraints. The overflow constraint and the non-negativity constraint both place lower bounds on production, while the current capacity constraint places an upper bound. If the current capacity constraint binds, then the other two constraints do not. The relationship between the overflow constraint and the non-negativity constraint is more complicated. If $R_t + w_t - \bar{R} > 0$, then the non-negativity constraint is non-binding. If $R_t + w_t - \bar{R} < 0$, then the overflow constraint is non-binding. In both these cases, the unmentioned constraint may or may not bind. However, if $R_t + w_t - \bar{R} = 0$, then it is possible for both constraints to bind simultaneously. In fact, if one of the constraints binds, then the other must bind at least weakly. Furthermore, in this case it is possible that both constraints bind strictly. We can see that Euler equation 10 depends only upon the sum $\theta_t + \delta_t$. Nonetheless, we will see that it matters which constraint binds.

2.1 A period t constraint binds

If one of the period t constraints binds, then behavior in period t is nailed down exactly by the need to satisfy that constraint. If the current capacity constraint binds, then all available water is used and $h_t = R_t + w_t$. In this case, $\frac{\partial \pi_t}{\partial h_t}$ is greater than the shadow price, and $\lambda_t > 0$ measures this disparity.

If the overflow constraint binds, then just sufficient water to avoid an overflow is used and $h_t = R_t + w_t - \bar{R}$. If the non-negativity constraint is the only constraint that binds, then $h_t = 0$. In either case, $\frac{\partial \pi_t}{\partial h_t}$ is less than the shadow price, and $\theta_t + \delta_t > 0$ measures the disparity.

Since the Hydro Plant has no choice about satisfying the constraints, the above conclusions hold no matter what else is going on. Hence, in all that follows, as we analyze other situations we are implicitly assuming that no period t constraint binds.

2.2 The period $t+1$ non-negativity constraint binds

We consider first the case in which the non-negativity constraint binds in period $t+1$. It is possible that the overflow constraint binds as well, but we know that the current capacity constraint does not bind. Hence, we might have the following as our period t and period $t+1$ Euler equations

$$\frac{\partial \pi_t}{\partial h_t} = r\left(\frac{\partial \pi_{t+1}}{\partial h_{t+1}} + \frac{\partial \pi_{t+1}}{\partial q_{t+1}} \cdot \frac{\partial Q_{t+1}}{\partial h_{t+1}} \cdot \frac{\partial H_{t+1}}{\partial R_{t+1}} + \delta_{t+1}\right) \tag{11}$$

$$\frac{\partial \pi_{t+1}}{\partial h_{t+1}} + \theta_{t+1} + \delta_{t+1} = r\left(\frac{\partial \pi_{t+2}}{\partial h_{t+2}} + \frac{\partial \pi_{t+2}}{\partial q_{t+2}} \cdot \frac{\partial Q_{t+2}}{\partial h_{t+2}} \cdot \frac{\partial H_{t+2}}{\partial R_{t+2}} + \delta_{t+2}\right) \tag{12}$$

We can combine these two equations to arrive at

$$\frac{\partial \pi_t}{\partial h_t} = r^2 \left(\frac{\partial \pi_{t+2}}{\partial h_{t+2}} + \frac{\partial \pi_{t+2}}{\partial q_{t+2}} \cdot \frac{\partial Q_{t+2}}{\partial h_{t+2}} \cdot \frac{\partial H_{t+2}}{\partial R_{t+2}} + \delta_{t+2} \right) - r \left(\frac{\partial \pi_{t+1}}{\partial q_{t+1}} \cdot \frac{\partial Q_{t+1}}{\partial h_{t+1}} \cdot \frac{\partial H_{t+1}}{\partial R_{t+1}} - \theta_{t+1} \right) \tag{13}$$

If we ignore the last term in Equation 13, then we would have the Euler equation for a problem in which Hydro was simply not allowed to produce electricity in period $t+1$. Whether or not Hydro is allowed to produce in period $t+1$ should make no difference, as he has no desire to do so. Hence, the last term in equation 13 must be zero. That is $\theta_{t+1} = \frac{\partial \pi_{t+1}}{\partial q_{t+1}} \cdot \frac{\partial Q_{t+1}}{\partial h_{t+1}} \cdot \frac{\partial H_{t+1}}{\partial R_{t+1}}$. Proceeding in this manner then, we can see that if the first period following period t in which the non-negativity constraint does not bind is period $t+k$, then output in period t is set based upon

$$\frac{\partial \pi_t}{\partial h_t} = r^k \left(\frac{\partial \pi_{t+k}}{\partial h_{t+k}} + \frac{\partial \pi_{t+k}}{\partial q_{t+k}} \cdot \frac{\partial Q_{t+k}}{\partial h_{t+k}} \cdot \frac{\partial H_{t+k}}{\partial R_{t+k}} \right) \tag{14}$$

We can see that the value of k in equation 14 does not make a qualitative difference. Hence, for the sake of exposition, the remainder of our discussion will be for the case in which $\delta_{t+1} = 0$.

2.3 The $t+1$ non-negativity constraint does not bind

We are now working with the assumption that every multiplier in the Euler Equation 10 are equal to zero. This leaves us with the following for a period t Euler Equation

$$\frac{\partial \pi_t}{\partial h_t} = r\left(\frac{\partial \pi_{t+1}}{\partial h_{t+1}} + \frac{\partial \pi_{t+1}}{\partial q_{t+1}} \cdot \frac{\partial Q_{t+1}}{\partial h_{t+1}} \cdot \frac{\partial H_{t+1}}{\partial R_{t+1}}\right) \tag{15}$$

The first term on the Right Hand Side of Equation 10 is the marginal revenue in period $t+1$. The second term is the strategic effect. It is the increase in profits which arises because of Thermal's decrease in output in anticipation of an increase in Hydro's output. The interesting part of that equation is the strategic effect, and in particular the value of $\frac{\partial H_{t+1}}{\partial R_{t+1}}$. If we have $\frac{\partial H_{t+1}}{\partial R_{t+1}} = 0$ then we have a solution which looks very much like the closed loop solution in CM. If $\frac{\partial H_{t+1}}{\partial R_{t+1}} = 1$, then the period t Euler equation would be essentially identical to the first order condition for the closed loop in CM's two period model. That is, period $t+1$ would be much like the end period in the closed loop solution of a finite horizon game.

Let us consider the possibility that either overflow or current capacity constraint binds in period $t+1$. If the overflow constraint binds (weakly,) then any addition water passed onto period $t+1$ must be used immediately. This holds even if the overflow and non-negativity constraint both hold. If the current capacity constraint binds strictly, then any small amount of water passed onto period $t+1$ will be used immediately because it has higher value in period $t+1$ than in later periods. This implies that in both cases $\frac{dH_{t+1}}{dR_{t+1}} = 1$.[1] Hence, if either the overflow or current capacity constraint binds in period $t+1$, then it is as if period $t+1$ were the last period in a finite horizon game.

Lets say that one of these constraints binds in period $t+1$, but no constraint binds in period t. We know that $\frac{dH_{t+1}}{dR_{t+1}} = 1$. What about $\frac{dH_t}{dR_t}$? Clearly this derivative is not equal to one or zero. If if were equal to one, then an exogenous increase in R_t would all be consumed in period t. If it were equal to zero, then any increase in R_t would be all consumed in period $t+1$. Either situation would throw the Euler Equation 15 out of balance. Hence $0 < \frac{dH_t}{dR_t} < 1$.

How do things change as the period in which the constraint binds moves further into the future? That is, lets say that the constraint binds in period $t+k$. How does $\frac{dH_t}{dR_t}$ change as we change the value of k?

3 The Steady Cycle

Our objective here is to argue that the stable cycle is the appropriate prediction, and then demonstrate that if one chooses the appropriate starting date, then one can duplicate the steady cycle with a finite horizon equilibrium. We compare a T period steady cycle with a T period finite horizon model. A key issue is the choice of start date for the

[1]Strictly speaking, we may only be able to say that the right hand derivative is equal to one. However, since we presume that $\frac{\partial Q_{t+1}}{\partial h_{t+1}} > 0$ it is the right hand derivative which matters.

finite horizon model. With this in mind, we need to define new summation notation. For $t_1, t_2 \in 1, ...T-1$, let

$$\overset{\circ}{\sum}\nolimits_{t=t_1}^{t_2} a_t = \begin{cases} \sum_{t=t_1}^{t_2} a_t & \text{if } t_1 \leq t_2 \\ \sum_{t=t_1}^{T-1} a_t + \sum_{t=0}^{t_2} a_t & \text{if } t_1 > t_2 \end{cases}$$

We will see that a key determinant of Hydro's behavior is the total amount of water available in a year. Consider a one year, T period, finite horizon model. For the moment, treat Hydro as a thermal producer with zero marginal costs. Doing so, makes this a totally static model in which each period is treated independently. Let w_t^0 denote Hydro's output in period t in this case, and $W^0 = \sum_{t=0}^{T} w_t^0$ denote the total energy output by Hydro in this case.

Let $\Delta(t, \tau) = \overset{\circ}{\sum}\nolimits_{s=\tau}^{t} (w_s - w_s^0)$. If a run starts in period τ, then R_τ denotes the amount of water at the end of period t if $h_s = w_s^0$.

Proposition 1 *A necessary, but not sufficient, condition for a open loop solution to replicate the steady cycle is that* $\exists \tau$ *such that* $0 \leq R_\tau + \Delta(t, \tau) < \bar{R}$ *and* $\bar{W} = W^0$.

Proof: If the overflow constraint binds (even weakly) in period $t+1$, then there is a strategic effect in period t. If the current capacity constraint binds strictly in period $t+1$, then there is a strategic effect in period t. Hence, for the steady cycle to look like the open loop solution, neither mentioned situation can arise once the steady cycle has been entered. Hence $\frac{\partial \pi_t}{\partial h_t} = r^k \frac{\partial \pi_{t+k}}{\partial h_{t+k}}$. Since we have an infinite horizon model, this holds in the limit as $k \to \infty$, which implies that $\frac{\partial \pi_t}{\partial h_t} = 0$. Consequently, $h_t = w_t^0$, and a total of W^0 is used each cycle. Hence, if $\bar{W} > W^0$ (resp. $\bar{W} < W_t^0$,) then the current capacity (resp. overflow) constraint will eventually strictly bind. The condition $0 \leq R_\tau + \Delta(t, \tau) < \bar{R}$ is exaclty the condition that it is possible to set $h_t = w_t^0$ without the current capacity constraint binding strictly or the overflow constraint binding. Hence, if it is not possible to satsify this condition, then there is a period in which there is a strategic effect. ♣

Here I need two examples one where it works, one where it does not.

To make the model more tractable, we assume that we can partition the year into a wet season and a dry season. During the wet season, the inflow of water is greater than would reasonably be used. During the dry seaons it is less than would reasonably be used. For simplicity we assume that periods $t_w = 0$ through $t_d - 1$ are the we season, and that t_d through period $T - 1$ are the dry season. We assume that during the wet season $w_t > a_t/b$ while in the dry season $w_t = 0$. (Note, a_t/b is the competitive output with zero marginal costs.)

Proposition 2 *Consider a steady cycle. If $\bar{W} > W^0$, then the overflow constraint binds in period $t_d - 1$. Once may duplicate the steady cycle with the closed loop solution for the run starting at $\tau = t_d$.*

4 Appendix: Multiple Thermal Plants

We show here that nothing is gained by using multiple thermal plants. For simplicity we will work with M symmetric (identical costs) thermal plants and ignore time subscripts. The problem for a thermal plant is essentially static, and they act in any period to set their marginal revenue equal to marginal cost. Let us denote a given firm $i's$ output as q^i, the total output of all thermal plants as $\mathcal{Q}$, and the output of the Hydro plants as $\mathcal{H}$. We set $\mathcal{Q}^{-i} = \mathcal{Q} - q^i$. The reaction function for firm i is found by setting marginal revenue to marginal cost

$$a - b(\mathcal{Q}^{-i} + \mathcal{H}) - 2bq^i = c + zq^i \tag{16}$$

Solving for q^i, we get the reaction function

$$q^i = \frac{a - c - b(\mathcal{Q}^{-i} + \mathcal{H})}{2b + z}$$

Of course this reaction function applies to all thermal plants. Hence, the symmetry of the problem assures that we can specify an 'aggregate reaction function' for the thermal sector, by recognizing that all the q^i will be equal. In particular,

$$\mathcal{Q} = M\left(\frac{a - c - b\mathcal{H}}{z + (M+1)b}\right)$$

One can choose ϕ and ξ such that a single firm with marginal costs $= \phi + \xi q$ will duplicate the behaviour of the above group of M firms. This requires only setting $\xi = \frac{z-(M-1)b}{M}$ and $\phi = a - \frac{(a-c)(\xi+2b)M}{z+(M+1)b}$. Now if we set $\hat{z} = \frac{z+(M+1)b}{M} - 2b$, then a market with one firm and marginal cost of $c + \hat{z}q^i$ exactly replicates a market with M firms and marginal costs of $c + \hat{z}q^i$.