Auctioning the Digital Dividend: A Model for Spectrum Auctions

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Abstract

We model a spectrum auction as a multi-unit auction where participants use the goods purchased to participate in a constrained, multi-product downstream market. We use dynamic programming techniques to numerically solve for the optimal bidding strategy for firms in a clock auction. Firms often value constraining competitor market power highly, and inefficient firms will often bid aggressively to minimise competition. Regulators concerned with revenue maximisation have strong incentives to encourage this behavior, capping more efficient firms or capping entrants to the market. In contrast, social welfare concerns suggest that allocating spectrum units may be more efficient than using an auction.

Keywords: Clock Auction, Spectrum Auction, Telecommunications Market, Equilibrium Bidding, Capacity Constraints

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1 Introduction

In most modern economies, the "Digital Dividend" represents a substantial technological and financial windfall. Developments in television broadcasting allow the transmission of digital video and sound which require only one-sixth the bandwidth of inferior quality analogue transmissions. Therefore, replacing analogue transmissions with digital transmissions frees up a substantial quantity of spectrum frequencies, which can be easily employed to transmit wireless data, as used by mobile phones, laptop computers, and other devices.

These unallocated new spectra present good opportunities for an economy in many ways. First, wireless telecommunications companies can expand their services. Firms that have access to the new spectrum can effectively provide superior data services relative to firms who do not. Second, consumers can enjoy a wider range of services potentially delivered in a more timely manner. Third, the government benefits from the unallocated spectra in two ways. On the one hand, the government, as an auctioneer, can earn revenue from the auction. On the other hand, the government, as a regulator, can affect the degree of competition in the telecommunications market in order to increase total surplus. A common policy tool adopted by governments to achieve social optimum is to conduct an auction with caps on firms to limit their winnings. In this way, the regulator can prevent one firm from winning all units and becoming a monopolist.

Spectrum auctions are potentially an efficient way to allocate the new spectra across firms. Spectrum auctions do not represent an entirely new phenomenon in most countries.[1] A spectrum auction is an example of a *multi-unit auction*. The auctioneer is selling a collection of relatively homogeneous goods to multiple firms.[2] If the number of firms in the auction is small, then this can result in participants winning multiple units and obtaining market power. Our aim in this paper is to examine the equilibrium properties of a Digital Dividend auction at which participants compete for market power in the telecommunications market. Our model consists of a downstream market, in which firms play a Cournot game with capacity constraints, similar to Laye and Laye (2008). Specifically, firms produce two goods: low and high data use plans and are constrained in their ability to produce data plans by the

[1] The United States began its auctions for spectrum licenses during the 1990s, and has since assessed their efficiency. For example, see Cramton (1997), Cramton (1998), Kwerel and Rosston (2000), and Bush (2010). Meanwhile, the GSM (second generation mobile telecommunication) and UMTS (third generation) auctions in Europe from 1999 to 2001 attracted a lot of attention from the public for their interesting outcomes. For example, see van Damme (2002), Klemperer (2005), and Grimm, Riedel, and Wolfstetter (2003). Economists have also been surprised by the huge revenues the British government realized from the sale of its 3G telecom licenses (Binmore and Klemperer (2002)).

[2] In reality, spectrum frequencies are not truly homogeneous because technology-dependent synergies to having access to adjacent frequencies exist, while regional standards, technological limitations, and device manufacturers' decisions may make certain segments of a given band more or less desirable.

amount of spectrum they have available. We assume that new (Digital Dividend) spectrum is required to produce high use data plans. Having projected their potential profits in the downstream market, firms enter in a simultaneous uniform-price clock auction to increase their production capacity. It is worth noting that while we consider the spectrum auction as the motivating example, our model can be generalized to investigate other problems where there is interaction between a downstream market and an upstream auction/competition.[3]

We solve the auction problem numerically using dynamic programming techniques that allow firms to bid strategically. We find many instances of firms following mixed strategies in their bidding. In these cases, our model generates *distributions* of allocations, profits, levels of social welfare, and revenue for the auctioneer. We consider several scenarios for market structure and investigate the equilibrium outcomes using these measures. Furthermore, our computational procedure also allows us to explore the way in which an auctioneer might impose caps on how many spectra individual participants can win. Our findings show that regulators concerned solely with total surplus may find allocating spectra by government policy to be more efficient than using a spectrum auction. In contrast, regulators concerned with revenue maximisation find that a capping structure that favours less efficient outcomes is an effective allocation mechanism.

Our work contributes to the literature in several ways. First, at multi-unit auctions, bidders are often assumed to have non-increasing marginal valuation (NIMV) for the goods in question.[4] This is a classic feature in economic theory making the models tractable. Unfortunately, this assumption is not necessarily valid in our case because marginal value of a spectrum unit may increase with a bidder's market power. In our model, units sold in the auction affect the production capacity. Therefore, winning more units not only increases a firm's production capacity, but also limits its competitors' capacities, which gives it market power. In such a case, the marginal value of an additional spectrum unit *may increase* as a firm gets closer to being a monopoly. This effect of market power on marginal valuations is also analyzed by Eső, Nocke, and White (2010). Assuming complete information, they model a downstream industry where firms compete to buy capacity in an upstream Vickrey-Clarke-Groves auction. Our approach is similar to theirs, but we allow firms to compete in a multi-product industry and adopt the simultaneous clock auction as the mechanism for spectrum allocation. While the clock auction is more applicable in practice, it can generate multiple equilibria. Therefore, we investigate the distribution of these equilibria in our paper.

Second, potential market power in the downstream market may be of concern to a benevo-

[3]Examples include procurement auctions (to obtain market access) for medical drugs in the third-world countries and competition by airlines for landing slots; see Eső, Nocke, and White (2010).

[4]For example, see Kastl (2011); Hortaçsu and McAdams (2010); Mishra and Parkes (2009); Blume, Heidhues, Lafky, and Münster (2009); Riedel and Wolfstetter (2006); Katzman (1999).

lent government who also happens to be the auctioneer. The theoretical economics literature suggests that license auctions should be analyzed in conjunction with the downstream market where bidders apply spectrum units to provide wireless services for customers.[5] Recent experiments by Offerman and Potter (2000) illustrate that a relationship exists between auctioning of entry licenses and market prices. As a result, bidders' valuations depend not only on their licensed units but also on whether the remaining units are obtained by someone else and who it will be. Similar to their work, we incorporate the downstream market into our auction model. However, firms in our model compete in the auction to gain access the market (licensing) and to increase their production capacity.

Finally, we model the government as both an auctioneer and a regulator, and examine how the government's objective function affects the optimal cap structure in the auction. A trade-off may exist between the total surplus and the auction revenue. In fact, even if an auction is efficient, it may result in an inefficient downstream market, where one firm wins most of the units. In this case, the government may act as a regulator and aim to maximize some combination of total surplus in the downstream market as well as the auction revenue. Dana and Spier (1994) also focus on a risk-neutral government's problem to choose who produces in a downstream market. We consider a more general objective function, and allow the government to be risk-neutral or risk-averse. Furthermore, we compute the optimal caps by using the means and covariance matrix of the total surplus and revenue, for each possible level of caps.

The layout of the remainder of this paper is as follows. Section 2 outlines our model for the downstream market and the auction itself. This section also discusses our solution method. Section 3 presents our results for uncapped auctions, while Section 4 explores the optimal capping decision for a regulator organizing a spectrum auction. Lastly, section 5 concludes.

2 Model

2.1 Downstream Market

We begin by describing the telecommunications market, which we will refer to as the downstream market throughout this paper. Suppose M firms operate in the market, which is composed of two products: high and low data use plans (henceforth, products). Let q_{hi} and q_{li} denote firm i's output for the high and low products. Cost of production may differ

[5] For example, see Janssen and Karamychev (2007); Hoppe, Jehiel, and Moldovanu (2006); Janssen (2006); Jehiel and Moldovanu (2003).

across firms as well as products. Thus, let $C_{hi}(q)$ and $C_{li}(q)$ represent firm i's total cost of producing q units of the respective products.

Inverse demand for the two products are described as follows:

$$\text{Low Product: } P_l = P_l\left(\mathbf{q_l}; \boldsymbol{\beta}_l\right) \quad (1)$$
$$\text{where } \mathbf{q_l} = (q_{l1}, q_{l2}, \ldots, q_{lM})$$
$$\text{High Product: } P_h = P_h\left(\mathbf{q_h}; \boldsymbol{\beta}_h\right) \quad (2)$$
$$\text{where } \mathbf{q_h} = (q_{h1}, q_{h2}, \ldots, q_{hM}).$$

In equations (1) and (2), $\mathbf{q_l}$ and $\mathbf{q_h}$ are the vector of outputs for each product by all firms, while $\boldsymbol{\beta}_l$ and $\boldsymbol{\beta}_h$ are the corresponding demand parameters. Note that if all firms' outputs for a product are perfect substitutes, then one can simplify the inverse demand functions so that price depends only on the aggregate production.[6]

We assume that firms are endowed with an initial allocation of legacy spectra, which we refer to as old spectra, and denote as $\mathbf{B_l}$. Meanwhile, depending on the auction outcome, firms may win units of the new spectra:

$$\text{Old Spectra: } \mathbf{B_l} = (B_{l1}, B_{l2}, \ldots, B_{lM}) \quad (3)$$
$$\text{New Spectra: } \mathbf{B_a} = (B_{a1}, B_{a2}, \ldots, B_{aM}). \quad (4)$$

Using the old and new spectra, firm i faces the following capacity constraints:

$$q_{hi} \leq \theta_{ai}\ B_{ai} \quad (5)$$
$$q_{hi} + q_{li} \leq \theta_{ai}\ B_{ai} + \theta_{li}\ B_{li} \quad (6)$$
$$q_{hi} \geq 0 \quad (7)$$
$$q_{li} \geq 0 \quad (8)$$

where firm i can use the new spectra to increase capacity for the high product, the low product, or both. In contrast, the legacy spectra only allow the firm to produce the low product. In the equations (8), θ_{ai} and θ_{li} represent the marginal increase in capacity for the old and new spectra, respectively. Finally, the last two inequalities ensure non-negative production for the two products.

[6] Alternatively, one could argue that the product prices may depend on both vectors of outputs, as the two products may be complements or substitutes. This relationship is easily incoporated into the demand functions.

We assume that the initial endowments by firms are already paid for or can be regarded as sunk costs. Given the spectrum allocations $\{\mathbf{B_l}, \mathbf{B_a}\}$, one can write down firm i's profit in the downstream market as follows:

$$\begin{aligned}\Pi_i^D(q_{hi}, q_{li}, B_{ai}, \mathbf{q_{h,-i}}, \mathbf{q_{l,-i}}, \mathbf{B_{a,-i}}, \mathbf{B_l}) =& P_h\left(\mathbf{q_h}; \boldsymbol{\beta}_h\right)\ q_{hi} - C_{hi}(q_{hi}) + \\ & P_l\left(\mathbf{q_l}; \boldsymbol{\beta}_l\right)\ q_{li} - C_{li}(q_{li}) \end{aligned} \tag{9a}$$

$$\begin{aligned}\Pi_i(q_{hi}, q_{li}, B_{ai}, \mathbf{q_{h,-i}}, \mathbf{q_{l,-i}}, \mathbf{B_{a,-i}}, \mathbf{B_l}, P_a) =& \Pi_i^D(q_{hi}, q_{li}, B_{ai}, \mathbf{q_{h,-i}}, \mathbf{q_{l,-i}}, \mathbf{B_{a,-i}}, \mathbf{B_l}) - \\ & C_a(B_{ai}, P_a)\end{aligned} \tag{9b}$$

where $\{\mathbf{q_{h,-i}}, \mathbf{q_{l,-i}}\}$ represents the vector of residual supply, and $\mathbf{B_{a,-i}}$ denotes the new spectra allocated for firm i. The first equation in (9a) is the firm's profit in the downstream market, whereas the firm's profit net of the cost of the new spectra is given in equation (9b). The right-hand side in equation (9a) is the sum of the profits from the high and low products, while the last term in equation (9b) is the cost of the new spectra, which depends on the auction price P_a.

Finally, the assumption below describes the information structure in the downstream market.

Assumption. *Assume that there is complete information in the market. In other words, all the information regarding the cost of production, the spectrum allocations, and the outputs by each firm, as well as the demand parameters are public information.*

Assuming complete information, firms can first solve for their profits in the downstream market, given the spectrum allocations. Then, they can enter the auction for the new spectra and use the information on their marginal valuations to formulate their bidding strategies.

2.2 Profit Maximization in Downstream Market

Given the spectrum allocations, we assume that the firms compete in a Cournot game by deciding how much to produce for both high and low products. Since firm outputs for the high and low products are limited by the capacity constraints described in equations (8), the distribution of the new and old spectra across firms can result in market power. Before we write down the profit maximization problem, we first describe the feasible sets for firm i:

$$\mathcal{A}_{1i} = \{(q_{hi}, q_{li})\ :\ (q_{hi}, q_{li}) \text{ satisfy capacity constraints in (8) given } \mathbf{B_a}\} \tag{10}$$

Using the feasible set defined above, firm i's profit maximization problem is as follows:

$$\max_{<(q_{hi},q_{li})\in\mathcal{A}_{1i}>} \Pi_i^D(q_{hi}, q_{li}, B_{ai}, \mathbf{q_{h,-i}}, \mathbf{q_{l,-i}}, \mathbf{B_{a,-i}}, \mathbf{B_l}) \qquad (P1_i)$$

where firm i chooses the outputs (q_{hi}, q_{li}) from the feasible set $\mathcal{A}_1$, conditional on the spectrum allocations. Therefore, we are interested in firms' production decisions assuming $\mathbf{B_a}$ is the result of the auction. In this way, we can derive the payoffs for firms, which we then use to solve for the auction equilibrium. We first define the equilibrium in the downstream market.

Definition 2.1. A Nash Equilibrium in pure strategies in the Cournot game for the downstream market is such that given $(\mathbf{B_l}, \mathbf{B_a})$:

$$\{q_{hi}^*, q_{li}^*\}_{i=1}^M \text{ solves } (P1_i); \forall\ i = 1, \ldots, M.$$

In other words, given the equilibrium behavior of all other firms $\left(\mathbf{q}^*_{\mathbf{h},-\mathbf{i}}, \mathbf{q}^*_{\mathbf{l},-\mathbf{i}}\right)$, firm i does not find it profitable to change its production decision. Furthermore, if we assume that demand and cost functions are differentiable, then we can solve for the Nash equilibrium using the following complementarity problem from $(P1_i)$:

$$\begin{aligned}
\frac{\partial \Pi_i^D(q_{hi}, q_{li}, \cdot)}{\partial q_{hi}} + \eta_i - (\lambda_i + \mu_i) \geq 0 \perp q_{hi} \geq 0; \forall i = 1, \ldots, M \\
\frac{\partial \Pi_i^D(q_{hi}, q_{li}, \cdot)}{\partial q_{li}} - \lambda_i \geq 0 \perp q_{li} \geq 0; \forall i = 1, \ldots, M \\
\theta_{ai}\ B_{ai} - q_{hi} \geq 0 \perp \mu_i \geq 0; \forall i = 1, \ldots, M \\
\theta_{ai}\ B_{ai} + \theta_{li}\ B_{li} - (q_{hi} + q_{li}) \geq 0 \perp \lambda_i \geq 0; \forall i = 1, \ldots, M \\
q_{hi} \geq 0 \perp \eta_i \geq 0; \forall i = 1, \ldots, M \\
q_{li} \geq 0 \perp \xi_i \geq 0; \forall i = 1, \ldots, M
\end{aligned} \qquad \text{(CP1)}$$

where $\{\mu_i, \lambda_i, \eta_i, \xi_i\}$ are the Lagrange multipliers on the capacity constraints given in equations (8). The complementarity problem (CP1) is a square problem with $6 \times M$ equations and $6 \times M$ unknowns $\{q_{hi}, q_{li}, \mu_i, \lambda_i, \eta_i, \xi_i\}_{i=1}^M$.

Remark. *The equilibrium for the capacity-constraint Cournot model in the downstream market is unique if the demands for both high and low products are linear and the cost functions are positive convex.*

As shown by Monderer and Shapley (1996), the equilibrium in these potential games may not be unique. However, Laye and Laye (2008) demonstrate the uniqueness of Cournot-Nash equilibrium in multi-market model with linear demands, positive convex production costs and closed convex sets of capacity constraints. For the remainder of the paper, we assume that linear demands for both high and low products and convex cost functions. We describe the solution to the complementary problem, and demonstrate that our problem has closed convex constraints in appendix A.1.

2.3 Clock Auction for the New Spectra

We adopt a clock auction in this paper. Suppose that there are N items to be sold. The auctioneer's goal is to find a price such that there is no excess demand for the goods. We assume that the goods are homogeneous so firms' bids consist of a quantity of spectrum at the current auction price, rather than binary bids for individual units of spectra. Let P_a represents the current price in the auction. A clock auction works in the following way:

1. The auctioneer starts P_a at zero.

2. At the current price P_a, firms submit their bids:

$$\mathbf{B_a}(P_a) = (B_{a1}(P_a), \ldots, B_{aM}(P_a)).$$

   Firms may respond to each others' bids at the ongoing price.

3. If there is excess demand at the current price, then the auctioneer increases the price:

$$\sum_{i=1}^{M} B_{ai}(P_a) > N \ \Rightarrow P_a \text{ increases}$$

4. The previous step is repeated until the auctioneer increases the price to P_a^* such that:

$$P_a^* = \inf \left\{ P_a \ : \ \sum_{i=1}^{M} B_{ai}(P_a) \leq N \right\}.$$

When the auction ends, the equilibrium vector of spectrum allocation is $\mathbf{B_a^*}$. Firm i pays the auction price P_a^* for each unit it wins. Thus, the total cost of the spectra to firm i is $P_a^* \ B_{ai}^*$. The total revenue collected by the auctioneer equals $P_a^* \ \sum_{i=1}^{M} B_{ai}^*$, which may be less than $P_a^* \ N$, if the auction ends with an excess supply.

2.4 Discussion

Before we solve the auction in section 2.5, we discuss three important features of a spectrum auction. Specifically, we discuss the effects of market power in the downstream market, spectrum as a scarce resource, and the non-increasing bid structure.

Degree of Competition in the Downstream Market Even though firms are endowed with legacy spectra, which they can use to produce the low product, only the new spectra can be employed to provide production capacity for the new product, as modeled in equation 8. Therefore, unless a firm wins new spectra at the auction, it cannot produce in the high use product market. Consequently, the auction outcome affects the degree of competition in the downstream market. For example, suppose a single firm wins all the units in the auction, so the other firms are forced out of the new-product market. In this case, the firm with all the spectra can operate as a monopolist in the market. This situation could occur if one of the firms is substantially more efficient in production, so that it faces much lower production costs. A contrasting situation is one in which all firms are relatively similar in terms of production costs. In this case, all firms are likely to win at least some portion of the new spectra, which makes the downstream market an oligopoly.

Potential market power affects the marginal value of the new spectrum. When there is no market power, the standard assumption is that an additional spectrum unit brings less and less profit to a firm. Therefore, the firm has a diminishing marginal valuation for spectrum. This would be the case if the firm has a concave profit as a function of the spectra won. However, in our case, profits are not necessarily concave in certain regions. As a graphical illustration, we depict a firm's profit as a function of the spectrum units in a two firm case in figure 1. In this example, the firm has some legacy spectra, so it generates profits despite losing all units at the auction. The different curves in figure 1 represents the firm's profits when the auction price increases. Holding the auction winnings constant, a higher auction price will lower firms' profits. The circles on the curves represent the optima of the profit functions at given auction prices, while the number of spectrum units available is 9.

In Figure 1, winning spectrum units has two distinct benefits to the firm. First, the more units won, the higher the capacity to produce the new product; therefore, profit goes up. However, this marginal increase in profit eventually declines to zero. Second, winning more units limits the competitor's capacity to produce the new product. In other words, the more units won by a firm, the greater the market power held by it. Consequently, the profit function is not concave after a certain level. As also noted by Eső et al. (2010), the first factor dominates for the first few units won (to enter the market), while the second factor is important for the later units won (to obtain market power). When the auction price is low, the firm aims to win all units to be a monopolist in the high use market. However, once the auction price rises above a certain threshold, this is no longer profitable. At this price or above, the firm shares the market with its competitors.

Spectrum as a Scarce Resource The number of available units in the spectrum auction is fixed at N units. This is important as it makes spectrum a scarce resource. In other words, the auctioneer (generally, the government) has a resource constraint. This resource constraint introduces a dynamic structure to our model in the following way: at any auction price P_a, firms aim to maximize their own profits. In doing so, firms face a trade-off. On the one hand, firms may keep their demand high to limit their opponents' production. If firms' bids lead to excess demand, the auction continues for at least another round, in which case the auction price goes up. On the other hand, if a firm drops its demand, the auction may clear before the price increases any further. Since the firms submit their bids simultaneously, each bidding decision of the auction is essentially a simultaneous finite state game, where firms evaluate their own profits in both markets and decide if they would like to bid more aggressively to constrain the other firms' production in the new product market, or to bid for only what they need to produce optimally. As a result, a dominant strategy may not exist, and the firms may randomize between allowing the auction to continue, and forcing the auction to conclude at the current price.

Non-Increasing Bids In a clock auction, bidders are often not allowed to increase their bid as the auction progresses. Given the dynamic setup for the auction, we update the action set for each firm depending on what they bid at the end of the previous round. This introduces history dependence, so firms' latest bids from the last round are included in the set of state variables of the dynamic problem.

2.5 Solving the Auction

We now demonstrate how to solve for the equilibrium in both markets, incorporating the rival nature of the spectrum and the fact that firms cannot increase their bids, which makes the dynamics of the auction important. We assume that the firms can only submit their bids in discrete amounts. We list the important rules in discrete bidding below:

1. In each round, firms submit their bids in discrete units:

$$\mathbf{B_a} = (B_{a1}, \ldots, B_{aM}) \in \{0, 1, 2, 3, \ldots, N\}^M .$$

2. At the end of each round of activity, if demand for units of spectrum exceeds the supply of spectra, the price of the units is increased by an increment ΔP.

3. Within each round, firms must decide whether to lower their reported demand or not. If multiple firms wish to lower their demand simultaneously, only one firm will succeed,

and firms are considered equally likely to succeed. We also allow sequential dropping; i.e., if a firm drops demand by one unit, then the firm has the option to drop its demand further at the same price. Note that once the firm stops dropping its demand, the other firms can react without experiencing an increase in the price.

4. A round of bidding only ends when all firms have chosen not to act. A firm who does not initially succeed in lowering its demand may try again prior to the auction price increasing.

Given our solution to the downstream market and profit structure given in equations (9), we can calculate the payoffs that any firm receives at the end of the auction. Suppose that the vector $\mathbf{B_a}$ is the firms' winnings at the end of the auction. Then, firm i's profit net of spectrum costs equals:

$$\begin{aligned}\bar{V}_i(\mathbf{B_a}, P_a) &= \Pi_i(q^*_{hi}, q^*_{li}, B_{ai}, \mathbf{q}^*_{\mathbf{h},-\mathbf{i}}, \mathbf{q}^*_{\mathbf{l},-\mathbf{i}}, \mathbf{B}_{\mathbf{a},-\mathbf{i}}, \mathbf{B_l}, P_a) \\ &= \Pi^D_i(q^*_{hi}, q^*_{li}, B_{ai}, \mathbf{q}^*_{\mathbf{h},-\mathbf{i}}, \mathbf{q}^*_{\mathbf{l},-\mathbf{i}}, \mathbf{B}_{\mathbf{a},-\mathbf{i}}, \mathbf{B_l}) - P_a\ B_{ai}\end{aligned} \tag{11}$$

where the first term on the equation is the downstream profits, while the second term is the cost of the spectrum items to firm i. We will refer to the value function in equation (11) as the *terminal value* for firm i.

With this structure in mind, we solve the auction as a dynamic programming problem using backward induction. We begin by considering a high level of price, and presume that at this stage, all firms would wish to exit the market. This results in all firms purchasing zero units of the new spectrum (and earning payoffs based upon their profits due to producing using their existing spectrum holdings). The market will thus clear at this stage.

Equilibrium in the auction is described by two sets of numbers. The first is the probability that each firm drops its demand, given a particular combination of competitors' demands, its own demand, and the current price. The second is the expected payoff for a firm as a function of the current set of firms' bids, the probability of dropping demand, and the current price:

$$\{V_i(\mathbf{B_a}, P_a, \boldsymbol{\pi}), \pi_i(\mathbf{B_a}, P_a)\}^M_{i=1}\,.$$

To calculate V_i and π_i, we proceed by backward induction, working back from the terminal (high) price of the auction. For each price level, we work through the different combinations of $\mathbf{B_a}$ sequentially, beginning with cases where $\sum_{i=1}^M B_{ai} = 1$, before proceeding to cases where $\sum_{i=1}^M B_{ai} = 2$, etc. First, we define the *continuation value* ($\hat{V}_i(B_a, P_a)$) which is attained if no firm reduces demand at the given price:

$$\widehat{V}_i(\mathbf{B_a}, P_a) = \begin{cases} V_i(\mathbf{B_a}, P_a + \Delta P) & \text{if } \sum_{i=1}^{M} B_{ai} > N \\ \bar{V}_i(\mathbf{B_a}, P_a) & \text{if } \sum_{i=1}^{M} B_{ai} \leq N. \end{cases} \tag{12}$$

In equation (12), the continuation value (see equation (11)) equals the terminal value, if there is no excess demand at the current price, in which case the auction ends. Otherwise, the auction continues with a higher price. Before we define the value function for firm i during the auction, we find it important to introduce one more notation: the change in firm i's expected payoff due to firm j dropping its demand by one unit $\Delta V_{ij}(\mathbf{B_a}, P_a)$ as:

$$\Delta V_{ij}(B_{aj}, \mathbf{B_{a,-j}}, P_a) = V_i(B_{aj} - 1, \mathbf{B_{a,-j}}, P_a) - \hat{V}_i(\mathbf{B_a}, P_a) \tag{13}$$

where $\Delta V_{ij}(\cdot)$ will only be well-defined for cases where $B_{aj} \geq 1$. Note that any reduction in demand results in the auction moving to a new level of demand, and that the price remains at the same level, which allows further reductions to take place.

Suppose that each firm chooses to drop its demand with probability π_i. Then firm i's expected payoff is given by:

$$V_i(\mathbf{B_a}, P_a, \boldsymbol{\pi}, \phi = 0) = \widehat{V}_i(\mathbf{B_a}, P_a) + \sum_{\substack{\delta_1, \ldots, \delta_M \in \{0,1\}^M \\ \delta_1 + \ldots + \delta_M \neq 0}} \left\{ \left[\prod_{k=1}^{M} \pi_k^{\delta_k} (1 - \pi_k)^{1-\delta_k} \right] \left[\frac{\sum_{j=1}^{M} \Delta V_{ij}(\mathbf{B_a}, P_a)\ \delta_j}{\sum_{j'=1}^{M} \delta_{j'}} \right] \right\} \tag{14}$$

where δ_i is a binary variable that represents whether firm i drops its demand or not. The parameter ϕ indicates which firm has dropped the last time. Therefore, the case that ϕ equals 0 implies that no firm has priority in lowering demand, so each firm that aims to drop demand is equally likely to do so successfully. In equation (14), the first term is the case where no firm drops demand, while the second term (starting with the summation) is the case where at least one firm drops demand. In particular, the first component in the summation is the probability of seeing a given combination of firms *trying* to drop their demand, while the final fraction weights the change in firm i's expected payoff from each demand reduction by the probability that each succeeds, conditional upon trying.

If firm m has just dropped its demand by 1 (and therefore can drop further units before its competitors can react), then firm i's current valuation is:

$$V_i(\mathbf{B_a}, P_a, \boldsymbol{\pi}, \phi = m) = \hat{V}_i(\mathbf{B_a}, P_a) + \pi_m \ \Delta V_{im}(\mathbf{B_a}, P_a) + (1 - \pi_m) \ \Big(V_i(\mathbf{B_a}, P_a, \boldsymbol{\pi}, \phi = 0) - \widehat{V}_i(\mathbf{B_a}, P_a)\Big) \tag{15}$$

where equation (14) can be seen to be a special case of (15) where we define $\pi_0 = 0$. Given the current valuation, if firm i follows a mixed strategy, then it is indifferent between lowering its own demand and maintaining it at the current level:[7]

$$V_i(\mathbf{B_a}, P_a, \pi_i = 1, \boldsymbol{\pi}_{-i}, \phi) = V_i(\mathbf{B_a}, P_a, \pi_i = 0, \boldsymbol{\pi}_{-i}, \phi); \forall i = 1, \ldots, M. \tag{16}$$

Next, we define the mixed-strategy Nash equilibrium using the system of equations (16).

Definition 2.2. Nash Equilibrium in mixed strategies for the downstream market and the auction with discrete bidding is such that given $(\mathbf{B_a}, \mathbf{B_l}, P_a)$, for each firm i, there exists a probability $\pi_i(\mathbf{B_a}, P_a)$ which maximizes firm i's expected payoff:

$$V_i(\mathbf{B_a}, P_a, \pi_i, \boldsymbol{\pi}_{-i}, \phi) \geq V_i(\mathbf{B_a}, P_a, \rho_i, \boldsymbol{\pi}_{-i}, \phi); \forall \rho_i \in [0, 1]; \forall i = 1, \ldots, M \tag{17}$$

Solving for the vector of probabilities implies solving a system of equations, each of which is represented by (17) with the inequality binding, for each of the firms who is following a mixed strategy.

3 Results

As a numerical example, we focus on a scenario whose parameters are given in Table 1. Some comments regarding this choice of numbers are in order. We have chosen to consider an auction with $M = 3$ firms over $N = 9$ units. Three firms are sufficient to avoid the singular feature of the two firm case: that spectrum not won by a firm is necessarily either won by its one competitor or unused. With three firms, a firm who does not buy a particular spectrum unit may be uncertain as to which of its competitors will win the unit. The nine-

[7]We solve this system of equations using Newton's method. It is helpful to note that

$$\frac{\partial V_i(\mathbf{B_a}, P_a, \boldsymbol{\pi}, \phi = 0)}{\partial \pi_n} = \sum_{\substack{\delta_1, \ldots, \delta_M \in \{0,1\}^M \\ \delta_1 + \ldots + \delta_M \neq 0}} (2\delta_n - 1) \left[\prod_{k \neq n} \pi_k^{\delta_k} (1 - \pi_k)^{1 - \delta_k} \right] \left[\frac{\sum_{j=1}^{M} \Delta V_{ij}(\mathbf{B_a}, P_a) \ \delta_j}{\sum_{j'=1}^{M} \delta'_j} \right].$$

unit auction is also interesting, because it avoids possible problems where the number of units is not divisible by the number of firms. With three symmetric firms, it is possible for each firm to win three units of spectrum. With three firms and 9 units, there are $1,000$ bid combinations at a given price. Given that we assume \$0.1 price increments from \$0 to \$7, this generates $70,000$ combinations and $280,000$ finite state games if we condition on who dropped their bid the last time. We explain our solution procedure in Appendix A.2.

With our set of base case parameters, we consider six scenarios that we believe provide some insight into the workings of a spectrum auction. Each scenario is a slight variation of the benchmark model, which assumes parameter values given in Table 1. These scenarios are summarised in Table 2. Scenario 0 is the benchmark model where all three firms are symmetric. Scenarios 1–2 explore floodable low-use markets, i.e. low product demand is infinitely elastic. In these scenarios, firms can make profits in the low-use market without worrying about rival production. We investigate the effect of differing marginal costs on equilibrium behaviour in scenarios 3–4. In particular, scenario 3 explores the asymmetric case where one firm is less efficient than its competitors. Scenario 4 considers the situation where one firm is more efficient than the others. We explore whether more efficient firms can obtain a greater market share through higher prices. Scenarios 5 and 6 study the extent to which existing constraints due to allocations of legacy spectrum influence the auction of the new spectrum. In scenario 5, we consider a market where two incumbents (firms 2 and 3) have ample existing spectrum, whereas firm 1 (a newer entrant) could use the new spectrum to not only supply high use customers, but also to expand its market share of the low-use market. Scenario 6 reverses the situation and explores the case where there is one incumbent competing with two entrants.

Note that in all cases where the firms are not symmetric, firm 1 is different from firms 2 and 3. In many of the tables which follow, equilibria which differ only because firms 2 and 3 have been reversed will be aggregated. Hence, when we report an outcome with allocation $(2, 2, 1)$, this represents the two outcomes: $(2, 2, 1)$ and $(2, 1, 2)$. For the cases of symmetric firms, we aggregate across all outcomes which achieve the same distribution across firms (so that $(1, 2, 2)$ includes $(2, 1, 2)$ and $(2, 2, 1)$).

Scenario 0 presents our base case: where all firms are symmetric, and both markets have downward sloping demand curves. Although the firms are symmetric, most of the auction outcomes are not symmetric, featuring one or two firms dominating the market. Despite this asymmetry of individual auction results, the overall probability distribution of allocations is symmetric: each firm is as likely to dominate the market, or to be a small player.

Table 3 illustrates the auction outcomes. There is excess supply in equilibrium. This is because the legacy spectrum is sufficient to allow the firms to produce their desired quantities in the low product market. In addition, the triopoly structure leads firms to cut back production.[8] The average units won by a firm is about 1.34, while the average auction price is $1.54 per unit.

Two points are worth noting. First, not all units are sold at the end of the auction. This is due to the imperfect market structure: firms compete in Cournot oligopoly markets. In fact, this result persists in almost all the scenarios we consider. Second, each equilibrium outcome is path dependent. To illustrate this, consider the equilibrium (2,1,1) at the price $1.1. In this scenario, all firms demand 9 units until the price reaches $1.1. At this point, one firm may lower demand (all three firms play mixed strategies for lowering demand). If this happens, it and one of the other two firms will concede the market by dropping their bids to 2 units. These two firms will then mix dropping or not dropping (so that one firm demands only one unit), and the firm whose demand remains at 2 will then follow a sequence of mixing games with the firm who did not initially lower demand (i.e. who still bids 9 units) to determine whether the auction concludes with both firms demanding 2 or the lower bidder only demanding one unit, while the higher bidder demands two. Throughout this process, there are two possible problems for the firms. The first problem occurs when total demand is less than 9 units; in this case, if two firms both fail to drop, the auction will clear, and some firms will be forced to buy units which they have no use for. The second problem occurs when demand is greater than 9 units; if two firms fail to drop, they may see the price increase, which will reduce their net payoffs from the auction.

If no firms lower demand at $1.1, then the auction price rises to $2 before a similar game is played, with similar auction outcomes. As the price rises further, subsequent opportunities to clear the market become available. However, once the price rises to $2.5, demand for units by the firms lowers, such that some allocations are duopoly outcomes where one firm wins 2 and another wins between 1 and 4 units. As the price rises higher still (to $3 per unit), the most likely outcome becomes a triopoly with each firm winning one unit. The reason for this type of behavior is because firms find it more profitable for their opponents to drop their demand. However, if their opponents don't drop, then they would rather lower their own demand. This generates mixed equilibria in the finite state games. Appendix A.3 shows two examples of games which illustrate this.

The gap between possible auction clearing prices (in this case, $1.1 and $2) explains the willingness of firms who have initially lowered their bids to lower further. If a firm who lowered its demand refuses to lower further, the price will increase substantially. Given that the firm already faces a disadvantage in subsequent bidding starting from a more constrained bid, the expected payoff from staying in the auction is substantially lower than what the

[8]A perfectly competitive market would produce 8 units of the high product.

firm will achieve by capitulating at a low price.

Scenarios 1 and 2 explore the possibility that the low-use market has infinitely elastic demand; see Tables 4 and 5. In these cases, firms can choose to use units of spectrum won in the auction to supply the low market, earning a profit of $1 per unit in scenario 1 and $2 per unit in scenario 2. This has two important effects in the auction. First, any unit can be profitably held as long as the auction clears at a price below $1 (scenario 1) or $2 (scenario 2). Second, even if the auction clearing price is above this level, firms who hold sub-optimally high numbers of units can mitigate their losses by using the units in the low-use market.

As mentioned above, firms use the new spectrum in the low product market if the auction price is less than $1 per unit. Looking at Table 3, we see that the lowest auction price is $1.1 per unit, so having a perfectly elastic demand for the low product market only leads to higher prices. The average price in scenario 1 is $1.69, compared to $1.54 in scenario 0, while the average number of units won by a firm is almost the same.

In Scenario 2, the outcomes are qualitatively different from Scenario 0. Here, at any price up to $2, all units can be profitably held. Further, the firms can potentially drop demand (and clear the market) even at a price of $0. The auction can clear at low price levels, where one firm drops demand to 2. The other two firms then play a sequence of mixed strategies, resulting in an auction outcome where one holds five units and the other two, or one holds four units and the other three.

In Scenario 2, it is highly likely that the auction continues rather than closing at a very low price. Because of the heavy use of mixed strategies that takes place in this scenario, there is a high probability that neither firm lowers demand. As a result, the process of clearing the auction may take several price ticks after the first firm lowers demand. This entails some risk for the other firms (who have not dropped demand). In addition, there is subsequent potential for a slow clearing of the auction (i.e. one in which several price ticks take place before demand falls below supply). The combination of these two effects discourages a firm who has once dropped its demand from staying in the market. Once the price passes above $1.1, firms are more willing to lower demand, and the firm who first lowers demand drops its demand to 5 units (rather than 2). All three firms follow mixed strategies, and a symmetric auction outcome of three units apiece can eventuate for higher prices.

Not surprisingly, the prospect of selling all nine units in scenario 2 is very lucrative for the government. In Table 5, we see that the average auction clearing price drops by 25 percent from $1.69 in scenario 1 to $1.19 in scenario 2. Therefore, firms find it profitable to compete less and use some of the new spectrum in the low product market. Note that the average number of units won by a firm increases to almost 3, so there is almost no excess supply after the auction. Consequently, the expected revenue increases to $10.73 in scenario

2 from \$6.66 in scenario 1.

To summarise, compared with the benchmark scenario, Scenarios 1 and 2 simulate a floodable low market, where firms can profitably utilise spectrum units without worrying about rival production decisions. The low-use market becomes a safe haven and two possible outcomes can occur. In scenario 1, clearing the auction at a price above \$1 results in a modest decline in demand (and lower profits for the firms than a low price clearing auction). However, in scenario 2, if the price *did* rise above \$2, the firms will face a drastic decline in profits. This provides a strong incentive to clear the market at a price below \$2, where all units are sold.

In Scenarios 3 and 4, we investigate how production inefficiency affects a firm's winnings from the auction.[9] Scenario 3 (Table 6) shows how two efficient firms compete in an auction with one less efficient firm (firm 1). Here, the auction almost always closes at a very low price: \$0.3. Equilibrium outcome $(1, 2, 2)$ occurs almost with probability 1. In this case, the two efficient firms let the inefficient firm stay in the market with one unit. Despite this, they enjoy a low auction price, which cuts down their costs, and obtain 2 units each, which increases their revenue. Scenario 3 features very low revenue for government, with a very low price clearing the auction.

Scenario 4 (Table 7) reverses the situation in Scenario 3. Here, firm 1 is more efficient than the other two firms. In scenario 4, firm 1 is more efficient than firms 2 and 3. However, firm 1 cannot impose dominance easily by increasing the auction price. This leads to two types of outcomes. First, firm 1 lets firms 2 or 3 win 2 units at a lower price. In the second outcome, firm 1 increases the price until both inefficient firms drop their demand to 1 unit. The average price stays low due to the lack of competition, so the government revenue is also low. These results stem from the uniform price auction format, since all units are sold for the same price, which makes it difficult to keep inefficient firms out of the market.

From the government's perspective, Scenario 4 is considerably more lucrative. This is partly due to the lower cost structure for firms in aggregate in Scenario 4, but also due to the auction outcomes. In both Scenarios 3 and 4, an equilibrium where two firms win 2 units and one firm wins 1 unit can eventuate. When one firm is less efficient, it is clear that this firm will be the smallest player in the market. However, when *two* firms are less efficient, competition between the two efficient firms to avoid the small position will drive up auction prices.

Scenarios 5 and 6 explore the ability of the new spectrum auction to mitigate or exacerbate spectrum disparities in the market; see Table 8 and 9. In scenario 5, firm 1 is a new

[9]Before we proceed with the results for these scenarios, it is worth noting that a firm's profit is not necessarily non-increasing in its own marginal cost. We illustrate this in Appendix A.3.

entrant to the market who has no units of legacy spectrum. In Scenario 5, the new entrant is unable to win more than one spectrum unit. This is because the two existing firms realise that allowing firm 1 to buy units not only deprives them of potential earnings in the high use data market (either by preventing their production or lowering the price received), but also potentially hurts their existing revenues from the low-use market. In effect, they derive more disutility from firm 1 winning units than firm 1 derives utility from winning.

Scenario 6 (Table 9) is the reverse of Scenario 5: in this scenario a single incumbent faces two potential entrants to the market, each having zero spectrum. Here the auction outcome is almost surely $(2, 1, 1)$ at an auction price of \$1.1 per unit. If the entrants attempt to stay in the market, the price will rise to \$1.6, and mixed strategies ensue. The entrants may be able to obtain a second unit as a result of this mixing (and the auction may end with the incumbent holding more than 2 units), however the expected payoff from this collection of auction outcomes is lower than receiving a single unit at price \$1.1 for the entrants (and is lower than receiving two units at \$1.1 for the incumbent) so the auction does not continue past \$1.1. This outcome (the incumbent managing to contain the winnings of the two entrants) is similar to Scenario 5. Again we see that the disutility for the incumbent from entrants gaining market share provides a strong incentive for the incumbent to dominate the auction.

4 Regulation through Caps

While an auctioneer generally aims to maximize revenue in a standard auction, a regulator tasked with running a spectrum auction may also take into account total surplus in the downstream market via the allocation of spectra. In particular, the regulator may face a trade-off between the auction revenue and the total surplus. For example, suppose that one of the firms is substantially more efficient. Then, this firm may increase the price high enough to win all the units. While selling all units at a high price is quite desirable as an auctioneer, the regulator may find out that the firm becomes a monopoly in the downstream market, which decreases the total surplus. As a result, the regulator may decide to intervene in the auction by setting *caps*, which stipulate the maximum number of units each firm can win in the auction.

Given that our solution technique works recursively backwards from high prices and low demand to low prices and high demand, we model caps by starting the auction with participants bidding demands that do not equal the total number of units available in the market but rather the cap set by the government. In fact, for each combination of starting bids, it is possible to evaluate the distribution of auction clearing bids. With this information available, the regulator optimally chooses a cap structure for the auction depending on his/her objective function, which we define as follows:

$$\max_{<\overline{\mathbf{B_a}} \geq 0>} \mathbb{E}\left(\phi_1\ TS + \phi_2\ R \mid \overline{\mathbf{B_a}}\right) + \phi_3\ \mathbb{V}\left[\phi_1\ TS + \phi_2\ R \mid \overline{\mathbf{B_a}}\right] \tag{18}$$

where $\overline{\mathbf{B_a}}$ denotes the vector of caps set by the regulator, and $\mathbb{E}(\cdot)$ and $\mathbb{V}(\cdot)$ represent the mean and the variance of the relevant variables, respectively. These caps limit how much each firm can win in the auction. The first two terms in equation (18) are the weighted average of the expected total surplus and revenue, and the last term is the variance of this weighted sum. This objective function generalizes the one used in Dana and Spier (1994), and allows the regulator to be risk-neutral (when $\phi_3 = 0$) or risk-averse (when $\phi_3 < 0$). For our numerical illustration, we consider five different types of regulators, and explore what the optimal cap structure for each regulator is for each of our scenarios:[10]

1. **Regulator 1:** This risk-neutral regulator is solely concerned with expected total surplus: the auction exists as a mechanism to ensure optimal allocation of spectra, and revenue is an unimportant side effect. In this case, the parameters of the social welfare function take values $(\phi_1, \phi_2, \phi_3) = (1, 0, 0)$.

2. **Regulator 2:** This risk-neutral regulator is only concerned with maximizing expected revenue. The parameters of the social welfare function take values $(\phi_1, \phi_2, \phi_3) = (0, 1, 0)$.

3. **Regulator 3:** This risk-neutral regulator maximizes the sum of the two.[11] In this case, $(\phi_1, \phi_2, \phi_3) = (1, 1, 0)$.

4. **Regulator 4:** A risk-averse regulator who seeks to maximize expected revenue less the variance of revenue. $(\phi_1, \phi_2, \phi_3) = (0, 1, -1)$.

5. **Regulator 5:** A risk-averse regulator who maximizes the sum of expected revenue and total surplus less the variance of revenue and total surplus. $(\phi_1, \phi_2, \phi_3) = (1, 1, -1)$.

[10] We find that regulators seeking to maximize expected total surplus choose cap structures which result in certain outcomes, so there would be no difference in the optimal cap structure chosen by a risk-averse total-surplus maximizer.

[11] A more libertarian view of the auction would be that government use of revenue earned might result in lower total surplus than would be achieved through this revenue being profits for the private sector. Our analysis could be extended to consider this by seeking to maximize total surplus with a penalty ascribed to revenue ($\phi_2 < 0$). However, for all our scenarios considered here, since the optimal strategy when maximizing total surplus almost always results in a collapse of the auction (see Table 10), a regulator with this viewpoint would choose a very similar cap structure to a regulator purely concerned with total surplus.

We display the results of our analysis in Table 10. Before explaining our results, three points are worth noting: First, we display the least restrictive cap among the ones that maximise the government objective function. Second, in the cases of symmetric firms, we only present one such cap structure, which appears asymmetric. For example, we present the (3,3,5) cap structure in scenario 0 for regulator 1, while (3,5,3), (3,3,5) and (5,3,3) all achieve the same outcome. Third, even though the optimal caps in each scenario may not appear to constrain the corresponding equilibria without caps, they affect the path to these equilibria. Therefore, some equilibria will not be reached since the initial condition (9,9,9) is no longer feasible.

Examining first the baseline Scenario (Scenario 0), we see several types of behavior. Regulator 1 finds it optimal to pick the winner for the auction. This results in a more competitive outcome (all firms receiving at least 2 units, with firms receiving at most 5 units) than the prevailing allocation from an uncapped auction (one firm winning 2 units and the other firms winning 1 unit). The trade-off with this approach is that there will be almost no auction activity, and the revenue will be zero. Regulators 2 and 3 run even auctions (in that all participants are equally capped) although the latter sets strictly lower caps than the former. This action raises the expected revenue above \$9 for Regulator 2 and \$6 for Regulator 3. This strategy is risky: Regulator 4 (who is risk averse) will hedge by lowering caps across the board to reduce variance while still generating revenue by having higher caps for *two* of the firms. Regulator 5 performs a riskless strategy like the risk-neutral benevolent Regulator (Regulator 1). Interestingly, the benevolent risk-neutral Regulator (1) has neither revenue nor risk in all Scenarios.

Scenarios 1 and 2 explore the floodable low-use market where there is infinitely-elastic demand. While Regulator 1 is keen to pick the winner, Regulator 2 sets equal caps for the three firms. In fact, the behavior of Regulator 1 and 2 remains consistent with Scenario 0: the benevolent Regulator opts to pick a winner and the revenue-focused Regulator runs a fair auction (with slightly tighter caps for the low revenue Scenario 2 and looser for the high revenue Scenario 3) to create a more competitive auction. The balanced Regulator (Regulator 3) tends to mix these behaviors: it runs *even* auctions for Scenario 0 and 2 while playing favourites in Scenario 1. Regulator 4 (a risk averse revenue maximiser) penalises one firm when the low market is floodable, even to the extreme of evicting this firm from the auction in the high revenue case (Scenario 2). Finally, Regulator 5 (risk-averse, but concerned with both revenue and welfare) chooses the same cap structure for Scenario 2 as Regulator 1 (resulting in no risk) but in Scenario 1 behaves the same as Regulator 3.

Turning our attention to Scenarios 3 and 4, we see different Regulators' attitude in the markets where firms are asymmetric due to their efficiency level. In Scenario 3 with firm

1 being *less* efficient than firms 2 and 3, some Regulators concerned with social surplus (Regulators 1 and 3) will punish firm 1. In contrast, Regulator 2 gives higher caps to the less efficient firm. This allows firm 1 to hold up the market, resulting in less efficient allocations of spectrum, but higher revenue for the Regulator. Regulator 5 chooses a similar approach to the revenue maximising firms, capping the more efficient firms more severely than firm 1. Note that helping the inefficient firm to hold up the market is generally a risky proposition, and Regulator 4 (who is risk-averse) chooses a riskless strategy by capping firm 1 somewhere between the other firms' caps. In Scenario 4 with firm 1 being *more* efficient than firms 2 and 3, all Regulators favour the efficient firm, in many cases not capping the firm at all, while using asymmetric caps on the inefficient firms. The only exception to this result is Regulator 1, who caps one inefficient firm at 3 and both other firms at 4.

When we explore the market with one entrant (who has no legacy spectrum) bidding against two incumbents, we see that Regulator 1 opts to favour the entrant over the two incumbents to encourage competition in the low-use market. In contrast, Regulator 2 favours the incumbents, encouraging them to run up prices and shut out the entrant (a similar holdup behavior to that exhibited by a less efficient firm). This is a risky strategy: if the Regulator is risk-averse (Regulator 4 or Regulator 5), a reverse strategy (that favours the entrant and treats the two incumbents asymmetrically) dominates. Regulator 3 sets a high cap for one incumbent, a medium cap for the entrant and a small cap for the other incumbent, encouraging entry, but also encouraging one of the incumbents to take a dominant position in the market. Examining the distribution of revenues and total surplus levels for these choices, we see that Regulator 5 chooses caps that lead to negative covariance between the two outcomes, effectively hedging his/her position by generating high social surplus in situations where revenue is low.

In Scenario 6, where firm 1 represents a single incumbent facing two entrants, Regulator 1 favours the entrants to the market. A *similar* behavior is exhibited by Regulators 2 and 4. Both risk-neutral and risk-averse revenue maximisers choose a much lower cap for the incumbent firm. Regulator 2 treats the two entrants equally, while Regulator 4 grants *one* entrant no cap, and the other a cap of 4, allowing the uncapped entrant considerable control over the auction's progress. In contrast, a Regulator whose objective is a balance between total surplus and revenue selects either an identical set of caps (Regulator 3) or a more generous cap for the incumbent (Regulator 5). Regulator 5's cap structure also results in neither social welfare risk nor revenue risk.

5 Conclusion

In this paper, we explore a multi-unit auction in which market participants use the auction units to provide services in a downstream market. Winning large quantities in the auction allows a participant to wield considerable power in the downstream market. We find that more efficient firms (or market entrants who value units of spectrum highly) are able to win more units than less efficient firms or incumbents. The presence of less efficient firms however, can result in small numbers of units being sold (a holdup problem). Regulators who are particularly concerned with revenue maximization will find favoring these participants in auction setups (with loose capping of their bidding) can result in sizable revenue capture. In contrast, in many cases, a social-welfare maximizing regulator will find that better outcomes can be achieved by simply assigning spectra to firms, than by allowing an auction to run.

With a general objective function for the government, we analyze the effect of spectrum caps in regulating auctions to enhance competition for wireless services. Our results for spectrum caps imply a similar argument to Hoppe, Jehiel, and Moldovanu (2006). The design where bidders are freely allowed to determine the number of spectrum units is unlikely to induce a market structure favorable to either social welfare or auctioneer's revenue. Therefore, regulator intervention by spectrum caps may sometimes be desirable.

Our model could explain the factors affecting the outcomes of the digital dividend auctions. A recent example is the Australian digital dividend auction that took place in May 2013. The Australian auction started with 9 units (as in our numerical examples) and ended at a low price equal to the reserve price ($311,067,000), after just the first round of bidding. We considered similar cases in scenarios 3 and 5 where one firm is either inefficient or has no legacy spectrum. In our examples, the auction cleared at a low price relative to the price in the benchmark scenario, and the two incumbents forced the inefficient firm (or the entrant) to win only a single unit.

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A Examples

A.1 Numerical Example: Linear Demand

Suppose that the demand for both products are linear and firms have quadratic costs:

$$
\begin{aligned}
P_h &= a_h - b_h\ Q_h = a_h - b_h\ \sum_{i=1}^{M} q_{hi} \\
P_l &= a_l - b_l\ Q_l = a_l - b_l\ \sum_{i=1}^{M} q_{li} \\
MC_{hi} &= c_{hi} + 2d_{hi}q_{hi}; \forall i = 1, \ldots, M \\
MC_{li} &= c_{li} + 2d_{li}q_{li}; \forall i = 1, \ldots, M \\
FC_i &= 0; \forall i = 1, \ldots, M
\end{aligned}
$$

where FC_i denotes the fixed cost of production for firm i, which is assumed to be zero. We can write the Lagrangian function as follows:

$$\begin{aligned} L_i =& P_h q_{hi} + P_l q_{li} - F_i - L_i - C_{li}(q_{li}) - C_{hi}(q_{hi}) + \\ & \lambda_i(\theta_{hi}B_{hi} + \theta_{li}B_{li} - q_{hi} - q_{li}) + \mu_i(\theta_{hi}B_{hi} - q_{hi}) + \eta_i q_{hi} + \zeta_i q_{li}. \end{aligned}$$

Therefore, the Cournot-Nash equilibrium (CNE) in the downstream market is the solution to the following problem where $\forall i = 1, \dots, M$:

$$P_h - b_h\ q_{hi} - (c_{hi} + 2d_{hi}q_{hi}) + \eta_i - (\lambda_i + \mu_i) \geq 0 \perp q_{hi} \geq 0 \tag{19a}$$

$$P_l - b_l\ q_{li} - (c_{li} + 2d_{li}q_{li}) + \xi_i - \lambda_i \geq 0 \perp q_{li} \geq 0 \tag{19b}$$

$$\theta_{ai}\ B_{ai} - q_{hi} \geq 0 \perp \mu_i \geq 0 \tag{19c}$$

$$\theta_{ai}\ B_{ai} + \theta_{li}\ B_{li} - (q_{hi} + q_{li}) \geq 0 \perp \lambda_i \geq 0 \tag{19d}$$

$$q_{hi} \geq 0 \perp \eta_i \geq 0 \tag{19e}$$

$$q_{li} \geq 0 \perp \xi_i \geq 0. \tag{19f}$$

Given the functional forms for the demand and costs, we can solve the complementary problem as a linear programming problem. More specifically, we can categorize the solution set as follows:

1. $\zeta_i = 0$ ($q_{hi} > 0$) and $\xi_i = 0$ ($q_{li} > 0$). This cases consists 4 sub-cases: $\lambda_i > 0$ and $\mu_i > 0$; $\lambda_i > 0$ and $\mu_i = 0$; $\lambda_i = 0$ and $\mu_i > 0$; $\lambda_i = 0$ and $\mu_i = 0$.

2. $\zeta_i > 0$ ($q_{hi} = 0$) and $\xi_i = 0$ ($q_{li} > 0$): in this case, $q_{hi} = 0 \leq \theta_{hi}B_{hi}$, hence $\mu_i = 0$. We consider two sub-cases: $\lambda_i > 0$ or $\lambda_i = 0$

3. $\zeta_i = 0$ ($q_{hi} > 0$) and $\xi_i > 0$ ($q_{li} = 0$). We consider two sub-cases: $\mu_i = 0$ or $\mu_i > 0$ as $q_{hi} \leq \theta_{hi}B_{hi} \leq \theta_{hi}B_{hi} + \theta_{li}B_{li}$, we have $\lambda_i = 0$ in both sub-cases.

4. $\zeta_i > 0$ ($q_{hi} = 0$) and $\xi_i > 0$ ($q_{li} = 0$). As $q_{hi} = 0 \leq \theta_{hi}B_{hi}$ and $q_{li} = 0 \leq \theta_{hi}B_{hi} + \theta_{li}Bli$, we have $\lambda_i = 0$ and $\mu_i = 0$.

Therefore, we have 9 cases in total. As mentioned above, we can solve each case as a linear programming problem.[12]

The sets of capacity constraints in our model are closed and convex, and hence the Cournot-Nash equilibrium in the downstream market is unique (see Laye and Laye (2008)).

[12] We refer the interested reader to our supplementary document, which is available on request from the authors.

To see this, let the set of capacity constraints faced by each firm i be $S_i = \{q_i \in R^2 / q_{hi} \geq 0; q_{li} \geq 0; q_{hi} \leq B_{ai}; q_{hi} + q_{li} \leq B_{ai} + B_{li}\}$. For all x and y in S_i and all t in $[0,1]$, consider $z = tx + (1-t)y$. z is non-negative and:

$$\begin{aligned} z_h &\equiv tx_h + (1-t)y_h \leq tB_a + (1-t)B_a = B_a \\ z_h + z_l &\equiv t(x_l + x_h) + (1-t)(y_l + y_h) \leq tB_l + (1-t)B_l = B_l \\ &\Rightarrow z \in S_i \end{aligned}$$

Thus S_i is a closed convex set.

A.2 Solving the Static Game

At each state, which is described by an auction price and a vector of current bids, each firm decides whether to drop its demand by one unit or not, given others' actions.Let F denote the set of players (firms) and each player has only two actions (drop and not drop). We describe the pseudo-algorithm below to solve the static games:

i) Check for active players: let F_0 denote the set of players with zero current bids. The set of active players is: $F_1 = F \backslash F_0$

ii) For each player in F_i, check if there is a dominant strategy. Let $\bar{F}$ denote the set of such players with a dominant strategy. The set of players who could play a mixed strategy is $F_2 = F_1 \backslash \bar{F}$.

iii) For the players in F_2, solve for the mixed-strategy Nash equilibrium (MSNE) in two ways:

   a) Root-finding method: Compute the roots of a system of non-linear equations, and check if the root (i.e. probability of dropping $\{\pi_i\}_{i \in F_2}$) satisfies the following condition: $\pi_i \in (0,1), \forall i \in F_2$

   b) Non-linear Optimization Method: If the root-finding method does not reveal a solution that satisfies the condition, then solve the nonlinear Nash equilibrium constrained optimization problem, with a predefined set of initial conditions.

iv) If the two methods in iii) do not yield an MSNE, look for pure-strategy Nash equilibrium (PSNE).

   a) If there is a single PSNE, then accept it as the solution to the game;

b) If there are multiple PSNEs, then we rank them according to their risk measure (Carlsson and van Damme (1993)):
$R_j = \prod_{i=1}^{F_2}(P_{s_{ij}} - P_{s'_{ij}})$ where s_{ij} is the strategy played in PSNE j and s'_{ij} is the one-step deviation from s_{ij} for player i. The risk mearure R_j for PSNE j is the product of the difference between equilibrium payoff ($P_{s_{ij}}$) and the corresponding one-step deviations ($P_{s'_{ij}}$). The PSNE with the highest risk measure is the solution to the game.

v) If no MSNE or PSNE are found in iii) or iv), then look for a semi-mixed solution where some players are constrained to play pure strategies and others are allowed to play mixed strategies. For each combination, we check whether the outcome is a Nash equilibrium, and assume that each occurs with equal probability.

A.3 An Example of Non-Monotonicity

Consider a two-player game where at some ongoing price, two firms each decide whether to drop their bid of one unit to zero units. The respective payoffs in each case are described as a 2x2 game in normal form:

Game 1:

	Firm 2 does not drop	Firm 2 drops
Firm 1 does not drop	1.62, 1.62	2.62, 1.92
Firm 1 drops	1.92, 2.62	2.27, 2.27

If both firms decide to drop, then a firm is randomly selected to drop first (with coin flip) so that the payoff for the strategy pair (drop,drop) is the average of the pairs (no drop, drop) and (drop, no drop). If neither firm drops, then the high-product market is not profitable at the dropping auction price ($1.62 for each firm). If only one firm chooses to drop, then firm's profit is derived solely from the low-product market, while the other firm enjoys being a monopoly in the high-product market, as well as earning low product profits.

It is clear that the two firms play a mixed-strategy Nash equilibrium in this game. The equilibrium probability of dropping demand is 0.4615 for both firms (since they are symmetric). The expected payoff in this game equals $2.0815 for each firm.

Game 2: Now, let firm 1 have a higher marginal cost for the high product (i.e., $MC_{h1} = 1.1$ and $MC_{h2} = 1$). The state-dependent payoffs for both firms are as follows:

	Firm 2 does not drop	Firm 2 drops
Firm 1 does not drop	1.52, 1.62	2.52, 1.92
Firm 1 drops	1.92, 2.62	2.22, 2.27

Note that firm 1's payoffs in each state are less than or equal to its payoffs in game 1. To have a mixed-strategy Nash equilibrium, firm 2 has to make firm 1 indifferent. Because firm 1's payoffs of dropping when it does not drop go down, firm 2 increases its probability of dropping. As a result, the equilibrium probability of dropping demand is (0.4615, 0.5714) for firms 1 and 2, respectively. The corresponding expected payoffs for firms 1 and 2 are (2.0914, 2.0815), which shows a *higher* expected payoff for firm 1 than before, despite the increase in its marginal cost.

B Tables

Symbol	Value(s)	Explanation
N	9	Number of units auctioned
M	3	Number of firms
a_h	9	Choke price in the new-product market
a_l	5	Choke price in the old-product market
b_h	1	slope of the demand curve for the new product
b_l	1	slope of the demand curve for the old product
$\mathbf{c}_h$	$(1,1,1)$	linear part of the cost of production of the new product
$\mathbf{c}_l$	$(1,1,1)$	linear part of the cost of production of the old product
$\mathbf{d}_h$	$(0,0,0)$	quadratic part of the cost of production of the new product
$\mathbf{d}_l$	$(0,0,0)$	quadratic part of the cost of production of the old product
$\boldsymbol{\theta}_h$	$(1,1,1)$	marginal increase in capacity for the new product
$\boldsymbol{\theta}_l$	$(1,1,1)$	marginal increase in capacity for the old product
$\mathbf{B}_l$	$(2,2,2)$	endowment of legacy spectrum
ΔP	0.1	price increment in auction

Table 1: Parameters for the benchmark model (see scenario 0 in table 3)

Note: All other scenarios are perturbations of these parameter values.

Scenario	Characteristics
0	Benchmark Model: All firms are symmetric.
1	Low market floodable: $a_l = 2$ and $b_l = 0$.
2	Low market floodable: $a_l = 3$ and $b_l = 0$.
3	Firm 1 is less efficient in high product market: $c_{h1} = 1.5$.
4	Firm 1 is more efficient in high product market: $c_{h1} = 0.5$.
5	Firm 1 has little legacy spectrum: $\mathbf{B}_l = (0,2,2)$.
6	Firms 2 and 3 have little legacy spectrum: $\mathbf{B}_l = (2,0,0)$.

Table 2: Scenarios

Note: Perturbations to the parameters outlined in table 1 to generate the scenarios for section 3. The parameter c_{xi} is firm i's linear part of the marginal cost of producing product x. The endowment for legacy spectrum is denoted by $\mathbf{B}_l$.

Probability	B_{a1}	B_{a2}	B_{a3}	P_a	TS	Rev
0.2032*	2	1	1	1.1	31.5	4.4
0.0708*	2	1	1	2	31.5	8
0.0246	1	1	1	3	27	9
0.0088*	2	1	1	2.5	31.5	10
0.0082*	2	1	1	3	31.5	12
0.0055**	2	1	0	3	27	9
0.0029*	2	1	1	1.2	31.5	4.8
0.0024*	2	2	0	3	31.5	12
0.0021*	2	1	1	2.1	31.5	8.4
0.0018	3	3	3	3	37.5	27
0.0018**	3	2	1	1.1	36.375	6.6
0.0017*	2	1	1	2.2	31.5	8.8
0.0013*	2	2	1	2	35	10
0.0012*	2	1	1	2.4	31.5	9.6
0.0011*	2	2	1	1.1	35	5.5
0.0009*	2	1	1	2.3	31.5	9.2
0.0006**	4	2	1	1.1	36.375	7.7
0.0004*	3	3	2	3	37.5	24
0.0004**	3	2	1	3	36.375	18
0.0004*	3	3	1	3	36.7778	21
Average	1.3409	1.3409	1.3409	1.5413	31.4038	6.1364

Table 3: Scenario 0/Benchmark model: All firms are symmetric.

Note: All three firms are as given in the base case (table 1). Each row represents a possible equilibrium for this auction. Columns represent (in order) probability of the equilibrium occurring, allocations to the three firms, price for the equilibrium in question, total surplus, and finally revenue. (*) indicates a representative equilibrium where there are total 3 symmetric cases. (**) indicates there are total 6 symmetric cases.

Probability	B_{a1}	B_{a2}	B_{a3}	P_a	TS	Rev
0.2283*	2	1	1	1.3	30	5.2
0.0583	1	1	1	3	25.5	9
0.0334*	2	1	1	2.1	30	8.4
0.0194*	2	1	1	3	30	12
0.0172*	2	1	1	2.3	30	9.2
0.0027*	2	2	1	2.3	33.5	11.5
0.0016**	2	1	0	3	25.5	9
0.0013*	2	2	0	3.3	30	13.2
0.0011*	2	2	1	2.1	33.5	10.5
0.0010*	2	2	0	3.1	30	12.4
0.0007**	2	1	0	3.3	25.5	9.9
0.0006*	2	2	0	3	30	12
0.0005	3	3	3	3	37.96875	27
0.0004**	3	2	1	2.3	34.5	13.8
0.0003*	2	1	1	2.2	30	8.8
0.0003**	2	1	0	3.1	25.5	9.3
0.0002	3	3	3	3.3	37.96875	29.7
0.0002*	3	3	2	3	36.96875	24
0.0002**	3	2	1	2.1	34.5	12.6
0.0001*	3	3	1	2.3	35.5	16.1
Average	1.3206	1.3206	1.3206	1.6998	29.7505	6.6645

Table 4: Scenario 1: Low market floodable: $a_l = 2.5$ and $b_l = 0$.

Note: All three firms are as given in the base case (table 1). Each row represents a possible equilibrium for this auction. Columns represent (in order) probability of the equilibrium occurring, allocations to the three firms, price for the equilibrium in question, total surplus, and finally revenue. (*) indicates a representative equilibrium where there are total 3 symmetric cases. (**) indicates there are total 6 symmetric cases.

Probability	B_{a1}	B_{a2}	B_{a3}	P_a	TS	Rev
0.0198**	4	3	2	1.5	46.875	13.5
0.0177*	5	2	2	1.2	46.875	10.8
0.0155*	5	2	2	1.3	46.875	11.7
0.0113*	5	2	2	1.4	46.875	12.6
0.0094*	5	2	2	1	46.875	9
0.0092*	5	2	2	1.1	46.875	9.9
0.0091**	4	3	2	1.3	46.875	11.7
0.0090*	5	2	2	0.9	46.875	8.1
0.0088**	4	3	2	1.2	46.875	10.8
0.0082**	4	3	2	1.6	46.875	14.4
0.0080*	5	2	2	0.8	46.875	7.2
0.0076**	4	3	2	1.4	46.875	12.6
0.0072*	5	2	2	1.5	46.875	13.5
0.0069**	4	3	2	1	46.875	9
0.0069**	4	3	2	1.1	46.875	9.9
0.0066*	5	2	2	0.7	46.875	6.3
0.0065**	4	3	2	0.9	46.875	8.1
0.0057**	4	3	2	0.8	46.875	7.2
0.0054**	4	3	2	1.7	46.875	15.3
0.0051*	5	2	2	0.6	46.875	5.4
Average	2.9988	2.9988	2.9988	1.1932	46.8681	10.7317

Table 5: Scenario 2: Low market floodable: $a_l = 3$ and $b_l = 0$.

Note: All three firms are as given in the base case (table 1). Each row represents a possible equilibrium for this auction. Columns represent (in order) probability of the equilibrium occurring, allocations to the three firms, price for the equilibrium in question, total surplus, and finally revenue. (*) indicates a representative equilibrium where there are total 3 symmetric cases. (**) indicates there are total 6 symmetric cases.

Probability	B_{a1}	B_{a2}	B_{a3}	P_a	TS	Rev
1	1	2	2	0.3	34.5	1.5
0	1	2	2	0.4	34.5	2
0	2	2	2	0.4	36.0938	2.4
0	2	3	2	0.4	36.3194	2.8
0	2	2	3	0.4	36.3194	2.8
0	2	2	2	0.5	36.0938	3
0	2	3	3	0.4	36.4297	3.2
0	2	4	3	0.4	36.4297	3.6
0	2	3	4	0.4	36.4297	3.6
0	2	4	2	0.4	36.3194	3.2
0	2	2	4	0.4	36.3194	3.2
0	2	2	3	0.5	36.3194	3.5
0	2	3	2	0.5	36.3194	3.5
0	2	3	3	0.5	36.4297	4
0	2	3	4	0.5	36.4297	4.5
0	2	4	3	0.5	36.4297	4.5
0	2	2	4	0.5	36.3194	4
0	2	4	2	0.5	36.3194	4
0	1	2	1	1.5	31	6
Average	1	2	2	0.3	34.5	1.5

Table 6: Scenario 3 Firm 1 is less efficient in high product market: $c_{h1} = 1.5$.

Note: All three firms are as given in the base case (table 1). Each row represents a possible equilibrium for this auction. Columns represent (in order) probability of the equilibrium occurring, allocations to the three firms, price for the equilibrium in question, total surplus, and finally revenue.

Probability	B_{a1}	B_{a2}	B_{a3}	P_a	TS	Rev
0.3333	2	2	1	0.6	36	3
0.3333	2	1	2	0.6	36	3
0.3333	2	1	1	1.6	32.5	6.4
0	2	1	1	1.7	32.5	6.8
Average	2	1.3333	1.3333	0.9333	34.8333	4.1333

Table 7: Scenario 4 Firm 1 is more efficient in high product market: $c_{h1} = 0.5$.

Note: All three firms are as given in the base case (table 1). Each row represents a possible equilibrium for this auction. Columns represent (in order) probability of the equilibrium occurring, allocations to the three firms, price for the equilibrium in question, total surplus, and finally revenue.

Probability	B_{a1}	B_{a2}	B_{a3}	P_a	TS	Rev
0.4560*	1	2	1	1.1	31.1111	4.4
0.0212*	1	3	2	1.1	35.9861	6.6
0.0134	1	2	2	1.1	34.6111	5.5
0.0077*	1	4	2	1.1	35.9861	7.7
0.0038*	1	2	1	1.2	31.1111	4.8
0.0030*	1	5	2	1.1	35.9861	8.8
0.0012*	1	6	2	1.1	35.9861	9.9
0.0002	1	2	2	1.2	34.6111	6
0.0001*	1	3	2	1.2	35.9861	7.2
0.0001*	1	2	1	1.3	31.1111	5.2
0.0000*	1	4	2	1.2	35.9861	8.4
0.0000*	1	5	2	1.2	35.9861	9.6
0.0000*	1	6	2	1.2	35.9861	10.8
0.0000	1	2	2	1.3	34.6111	6.5
0.0000*	1	2	1	1.4	31.1111	5.6
0.0000*	1	3	2	1.3	35.9861	7.8
0.0000*	1	4	2	1.3	35.9861	9.1
0.0000*	1	5	2	1.3	35.9861	10.4
0.0000	1	2	2	1.4	34.6111	7
Average	1.0000	1.5908	1.5908	1.1008	31.4833	4.6033

Table 8: Scenario 5 Firm 1 has zero legacy spectrum: $\mathbf{B}_l = (0, 2, 2)$.

Note: All three firms are as given in the base case (table 1). Each row represents a possible equilibrium for this auction. Columns represent (in order) probability of the equilibrium occurring, allocations to the three firms, price for the equilibrium in question, total surplus, and finally revenue. This case exhibits many equilibria, ranging from an early clearing of the market, where one competitor receives a small share, to a high revenue scenario where all spectrum is purchased. (*) notes that there are total two equilibria in this symmetric case, corresponding to two symmetric incumbent firms.

Probability	B_{a1}	B_{a2}	B_{a3}	P_a	TS	Rev
1	2	1	1	1.1	30	4.4
0	2	2	1	1.6	32.878	8
0	2	1	2	1.6	32.878	8
0	3	2	1	1.6	34.688	9.6
0	3	1	2	1.6	34.688	9.6
0	2	2	1	1.7	32.878	8.5
0	2	1	2	1.7	32.878	8.5
0	4	2	1	1.6	34.688	11.2
0	4	1	2	1.6	34.688	11.2
0	3	2	1	1.7	34.688	10.2
0	3	1	2	1.7	34.688	10.2
0	2	2	1	1.8	32.878	9
0	2	1	2	1.8	32.878	9
0	5	2	1	1.6	34.688	12.8
0	5	1	2	1.6	34.688	12.8
0	6	2	1	1.6	34.688	14.4
0	6	1	2	1.6	34.688	14.4
0	4	2	1	1.7	34.688	11.9
0	4	1	2	1.7	34.688	11.9
Average	2	1	1	1.1	30	4.4

Table 9: Scenario 6 Firms 2 and 3 have zero legacy spectrum: $\mathbf{B}_l = (2, 0, 0)$.

Note: All three firms are as given in the base case (table 1). Each row represents a possible equilibrium for this auction. Columns represent (in order) probability of the equilibrium occurring, allocations to the three firms, price for the equilibrium in question, total surplus, and finally revenue. This case exhibits many equilibria, ranging from an early clearing of the market, where one competitor receives a small share, to a high revenue scenario where all spectrum is purchased.

	Regulator 1						
	S0	S1	S2	S3	S4	S5	S6
Caps	(3,3,5)	(2,2,6)	(2,2,6)	(2,3,4)	(4,3,4)	(6,3,3)	(2,6,6)
$\mathbb{E}(TS)$	37.5000	37.9688	46.8750	36.4297	38.9297	37.5000	37.5000
σ^2_{TS}	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000*
$\mathbb{E}(R)$	0.0000*	0.0000	0.0000	0.0000	0.0000*	0.0000*	0.0000
σ^2_R	0.0000*	0.0000	0.0000	0.0000	0.0000*	0.0000*	0.0000
$\sigma_{TS,R}$	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000*	0.0000
	Regulator 2						
	S0	S1	S2	S3	S4	S5	S6
Caps	(8,8,8)	(6,6,6)	(9,9,9)	(6,2,2)	(9,4,8)	(4,6,6)	(2,7,7)
$\mathbb{E}(TS)$	30.9844	32.0205	46.8681	31.8789	30.2995	30.9608	26.2075
σ^2_{TS}	3.8172	19.5496	0.0318	2.9879	8.6615	3.4400	5.5186
$\mathbb{E}(R)$	9.0269	8.0148	10.7317	4.8160	9.1755	7.8499	9.7851
σ^2_R	5.1794	10.1160	11.7273	6.4868	2.9559	7.4259	6.6630
$\sigma_{TS,R}$	1.6318	-7.8118	-0.0186	-3.3814	4.6883	0.8789	4.6596
	Regulator 3						
	S0	S1	S2	S3	S4	S5	S6
Caps	(5,5,5)	(5,5,7)	(9,9,9)	(2,3,7)	(9,4,8)	(4,2,5)	(4,4,4)
$\mathbb{E}(TS)$	34.0553	37.5000	46.8681	34.7637	30.2995	35.1001	36.2523
σ^2_{TS}	6.6370	0.0000	0.0318	2.4838	8.6615	0.6407	0.3424
$\mathbb{E}(R)$	6.3682	4.6690	10.7317	2.4475	9.1755	3.9192	3.5249
σ^2_R	5.2054	0.1819	11.7273	2.9678	2.9559	0.2516	3.3932
$\sigma_{TS,R}$	-0.8011	0.0000	-0.0186	-1.3197	4.6883	0.3836	0.0709
	Regulator 4						
	S0	S1	S2	S3	S4	S5	S6
Caps	(4,6,6)	(8,9,9)	(0,9,9)	(5,3,9)	(9,4,8)	(8,3,5)	(2,4,9)
$\mathbb{E}(TS)$	32.0550	30.0000	45.9999	31.0000	30.2995	31.1111	32.1531
σ^2_{TS}	1.9205	0.0000	0.0009	0.0000	8.6615	0.0000	0.0000
$\mathbb{E}(R)$	6.3679	5.2000	9.4236	4.4000	9.1755	5.6000	5.0000
σ^2_R	0.7826	0.0000	9.3602	0.0000	2.9559	0.0000	0.0000
$\sigma_{TS,R}$	1.1191	0.0000	-0.0003	0.0000	4.6883	0.0000	0.0000
	Regulator 5						
	S0	S1	S2	S3	S4	S5	S6
Caps	(3,3,5)	(5,5,7)	(2,2,6)	(4,3,3)	(9,3,5)	(6,3,4)	(5,4,4)
$\mathbb{E}(TS)$	37.5000	37.5000	46.8750	36.3369	36.0000	36.7667	34.8000
σ^2_{TS}	0.0000	0.0000	0.0000	0.0175	0.0000	0.2689	0.0000
$\mathbb{E}(R)$	0.0000	4.6690	0.0000	0.3364	3.0000	1.0333	3.0000
σ^2_R	0.0000	0.1819	0.0000	0.2482	0.0000	0.0556	0.0000
$\sigma_{TS,R}$	0.0000	0.0000	0.0000	-0.0627	0.0000	-0.1222	0.0000
	Uncapped						
	S0	S1	S2	S3	S4	S5	S6
Caps	(9,9,9)	(9,9,9)	(9,9,9)	(9,9,9)	(9,9,9)	(9,9,9)	(9,9,9)
$\mathbb{E}(TS)$	31.4038	29.7505	46.8681	34.5000	34.8333	31.4833	30.0000
σ^2_{TS}	1.9412	1.8741	0.0318	0.0000	2.7222	1.6104	0.0000
$\mathbb{E}(R)$	6.1364	6.6645	10.7317	1.5000	4.1333	4.6033	4.4000
σ^2_R	6.8627	6.0918	11.7273	0.0000	2.5689	0.5439	0.0000
$\sigma_{TS,R}$	0.1327	-0.0456	-0.0186	0.0000	-2.6444	0.8791	0.0000

Table 10: Optimal capping strategies for different auctions

Note: Scenarios (S0–S9) are as listed in table 2. For each regulator, we display the capping structure (Caps), expected total surplus $\mathbb{E}(TS)$, expected revenue $\mathbb{E}(R)$, and the covariances of the variables $\{\sigma^2_{TS}, \sigma^2_R, \sigma_{TS,R}\}$. (*) notes positive numbers slightly above 0, which makes these caps appealing than other cases generating the same $\mathbb{E}(TS)$. The last blocks shows the outcome if there are no caps.

C Figures

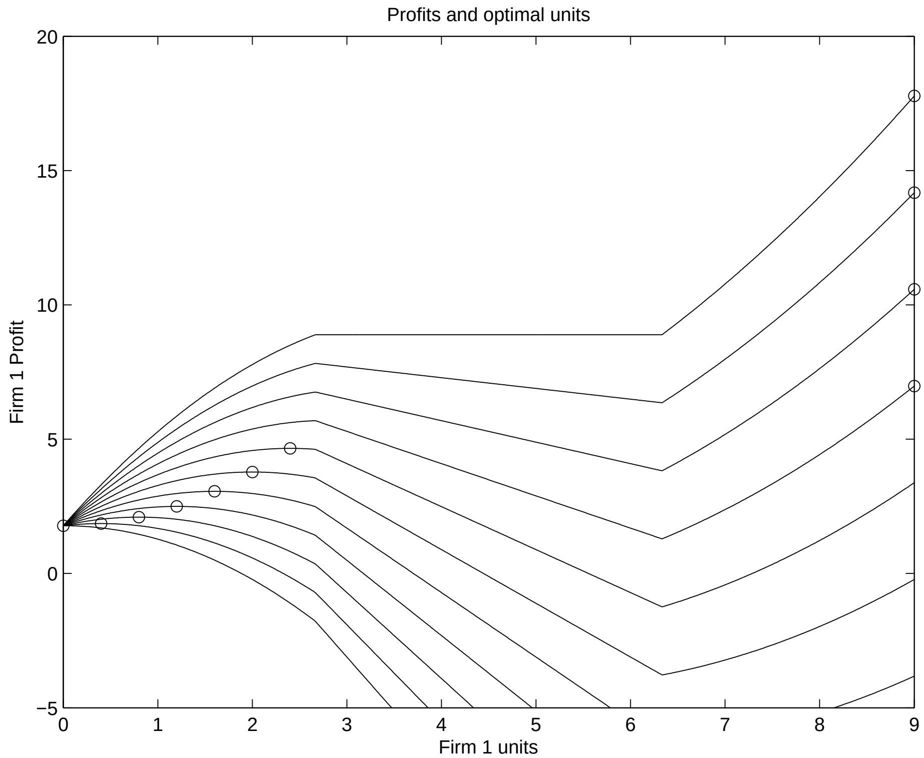

Figure 1: Duopoly Case: Profits vs. Auction Winnings

Note: These results assume market parameters as given in Table 1, with the exception of M (the number of firms) which is assumed to be two. The horizontal axis displays number of units firm 1 wins. Firm 2 is assumed to win the remaining units. Vertical axis displays firm 1's profit.