Optimal monetary policy and the exchange rate

[Incomplete draft, not for quotation]

James Graham Christie Smith

June 21, 2013

Abstract

Many papers suggest that 'optimal' monetary policy rules for small open economies should not respond to the exchange rate. However, these papers often neglect the link between the frictions present in the model economy and the welfare losses that households actually experience. We show that with imperfect pass-through of the exchange rate to (retail) import prices and uncovered interest parity shocks, law of one price and consumption gaps affect welfare and thus feature in the loss function that should be used to analyze optimal monetary policy. These terms are in addition to the domestic inflation and output gap terms that commonly feature in quadratic approximations to welfare. We then optimize the coefficients of a generalized Taylor rule conditional on a model estimated using Bayesian methods, and consider the implications of this rule for policy and the dynamics of the exchange rate.

1 Introduction

In this paper we use quadratic approximations to welfare to optimize an empirical monetary policy rule for a small open economy.[1] Our empirical model has both domestic *and* international frictions that monetary policy might reasonably seek to offset, and we evaluate the policy tradeoffs that arise from these multiple distortions. In particular, we identify the relative importance of exchange rate stability for the monetary authority.

The weights of the loss function, and hence the relative importance of different frictions, depend on the structural features of the economy. For example, using a calibrated model De Paoli (2009a) shows that the optimal monetary policy rule depends critically on the intertemporal and intratemporal substitution elasticities.[2] To inform practical policy-making, we estimate the model using Bayesian methods to identify the loss function that should be optimized via our monetary policy rule.

Our welfare approximation explicitly depends on the theoretical distortions in our model, and connects welfare to the utility of the representative agent, rather than positing a separate, possibly-arbitrary, loss function for the monetary authority. While conceptually straightforward, the computation of this

[1] See for example Rotemberg and Woodford (1997), Woodford (2003), Benigno and Woodford (2005) and many others.

[2] In a closely related paper De Paoli (2009b) shows that the structure of financial markets – whether they are complete or incomplete – also affects the ranking of policy rules.

second order approximation is quite involved given the multiple imperfections and shocks that we consider in our model.

Clarida et al. (2001) and Galí and Monacelli (2005) show that it is possible to establish an isomorphism between optimal monetary policy in open and closed economies, such that monetary authorities in both types of economy should simply respond to developments in an output gap and inflation. However, it is well-understood that the isomorphism depends on parametric assumptions embedded in the canonical small open economy model. Parameter estimates from quantitative studies imply that some of these restrictions (on the intertemporal and intratemporal substitution elasticities for example) do not hold, see for example Justiniano and Preston (2010b). Furthermore, Justiniano and Preston also estimate that the home and foreign goods sectors exhibit different degrees of price stickiness.[3] To provide a more compelling guide to optimal policy, we relax the parameter assumptions embedded in e.g. Galí and Monacelli (2005) and derive a welfare function that is consistent with empirically estimated parameter values.

Theoretical papers show that when there are terms of trade externalities, home bias, imperfect international risk sharing arrangements, or imperfect exchange rate pass-through into import prices, optimal policy should also take into account movements in the terms of trade or the exchange rate in addition to inflation and the output gap; see for example Corsetti and Pesenti (2001), Sutherland (2002), Benigno and Benigno (2003), Monacelli (2005), Corsetti and Pesenti (2005), Kirsanova et al. (2006), Faia and Monacelli (2008), De Paoli (2009b), and Engel (2009). Our approximation to welfare takes into account deviations from imperfect international risk sharing, as in Kirsanova et al. (2006), and the law of one price (loop) gaps that arise from imperfect exchange rate pass-through, as in Monacelli (2005). Given the empirical parameterization of our model the terms of trade externality is operative, and consumers also suffer from home bias.

Our model takes into account both the empirical failure of uncovered interest parity (UIP) and the failure of the law of one price.[4] We use shocks to international risk sharing to model the discrepancy in uncovered interest parity and we account for imperfect pass-through via domestic nominal rigidities in import prices.[5]

Monetary policy can ameliorate international distortions and imperfections by altering interest rates and hence the nominal exchange rate. Our optimal rule trades off stabilization of domestic frictions, such as conventional New Keynesian price rigidities, with the international frictions described above. While other policies, such as taxes or subsidies, could be used to offset the distortions considered here, we specifically wish to understand how monetary policy should best be amended to deal with these competing distortions. Central bank independence provides monetary authorities with a wide degree of latitude to

[3] Likewise, many theoretical models (see eg Corsetti and Pesenti 2001) make assumptions about the intratemporal substitution elasticity that ensure that trade balances are continuously zero, which is clearly inconsistent with empirical evidence for both large and small economies.

[4] See for example Froot and Thaler (1990), Engel (1996), Sarno (2005), Chinn (2006) on the failure of UIP and Rogoff (1996) and Campa and Goldberg (2005)on the failure of the law of one price.

[5] Engel (2000) discusses local currency pricing in Europe, and its implications for optimal currency areas.

alter their policy rules and such changes are, in a sense, practical to implement, whereas introducing taxes or subsidies is likely to be contentious and subject to the vagaries of the political process. See the references cited in De Paoli (2009a) for a discussion of alternative policy instruments.

Our analysis enables us to understand the extent to which optimal monetary policy deviates from actual policy. A number of authors have estimated generalized Taylor rules for small open economies, see e.g. Lubik and Schorfheide (2007), Justiniano and Preston (2010a,b). In the latter two references, the estimated rules imply that Australia and New Zealand respond very weakly to the change in the exchange rate (with coefficients somewhere between 0.03 and 0.21), while the Canadian policy response is somewhat stronger (with a parameter between 0.2 and 0.42).[6] Conditional on our representation of the international economy, we wish to understand whether *estimated* policy rules materially diverge from *optimal* policy rules.[7]

Following Justiniano and Preston (2010a) and others, we determine the optimal parameters of a generalized Taylor rule by minimizing the loss function subject to the estimated model equations.[8] The generalized Taylor rule that we optimize has the usual inflation and output arguments and lagged interest rate term, and is further augmented with the change in the exchange rate. This rule encompasses both the canonical Taylor rule from a closed economy (as when the monetary authorities only respond to inflation and output, with a coefficient of zero on the change in the nominal exchange rate, Δe_t) and a fixed exchange rate regime (as occurs when the coefficient on Δe_t is very large).

Policy rules with a differenced exchange rate argument have been used to model monetary regimes with 'target zones' for exchange rates (e.g. Svensson 1994) and to model the Swedish transition from a fixed exchange rate regime to a floating one (Adolfson et al., 2008). This type of rule introduces a tradeoff between interest rate variability and exchange rate variability.

Justiniano and Preston (2010b) explicitly analyze the optimality of different policy rules. They find that optimal policies do not respond to the nominal exchange rate, irrespective of whether parameter uncertainty is taken into account. Foreign shocks often play a relatively small role in the domestic fluctuations of small open economies in DSGE models (Justiniano and Preston, 2010a), though they may play a large role in driving the exchange rate. Since the loss function in Justiniano and Preston (2010b) contains only output and inflation, responding to the exchange rate implies responding to shocks that have little influence on the domestic fluctuations that alter welfare. Thus, as Justiniano and Preston (2010b) note, it is unsurprising that (given the parameterization of their model and the loss functions that they use to evaluate policy) their optimal policy implies little reaction to the exchange rate.

However, the class of loss functions that Justiniano and Preston optimize

[6] In Lubik and Schorfheide (2007) the 90 percent confidence interval for the response of UK monetary authorities to the change in the exchange rate is $[0.07, 0.19]$.

[7] In a similar vein, Kam et al. (2009) estimate the *loss* functions that underpin the policy rules of monetary authorities in small open economies. In Kam et al. (2009), the estimated loss functions describing monetary authorities' preferences imply no concern for stabilizing the real exchange rate in Australia, Canada and New Zealand. However, the loss functions they estimate are independent of the frictions that would micro-found policy responses to exchange rates.

[8] The simple rules that we explore provide more readily digestible guidance for policy-makers than, say, Ramsey policy.

is not directly derived from the welfare losses that arise from the particular inefficiencies in their model. In particular, their loss function only includes terms in output, inflation, and the interest rate, despite the fact that their model includes habit formation, inflation indexation, incomplete asset markets, preference shocks, and incomplete exchange rate pass-through. These features introduce additional arguments in quadratic approximations of the welfare function. In this paper we explore the relative importance of these model features for exchange rate stability. (Noting, however, that *estimated* rules suggest that policy-makers do not respond strongly to changes in the exchange rate.)

We also examine how optimal policy rules affect macroeconomic dynamics, relative to estimated policy rules. In particular, we explicitly investigate the extent to which an optimal policy rule would alter exchange rate dynamics, since exchange rate dynamics are often central to debate about the 'appropriateness' of monetary policy in small open economies. There is often considerable concern that 'non-fundamental' shocks might distort the value of the exchange rate, with attendant implications for the allocation of real resources across tradable and non-tradable sectors of the economy.

Using the structure of the model we can identify the shocks that are perturbing exchange rate dynamics, and can explore their contribution to historical macroeconomic outcomes.

The rest of the paper is organized as follows. Section 2 describes the model, and the derivation of the second order approximation. The particular frictions in our model are discussed in section 3. The welfare function is then discussed in section 4. The estimation is reported in section 5, and the optimal policy results are reported in section 6. Section 7 concludes.

2 The Model

Our model is similar to the New Keynesian small open economy model in Galí and Monacelli (2005) and Monacelli (2005). The model has four agents: households, firms, retailers, and a monetary authority. Households consume and provide labour; firms produce goods; retailers sell imported goods to domestic residents; and the monetary authority sets interest rates, aiming to stabilize the domestic economy. We now describe the decision problems of the agents in turn. The model is similar in most respects to Galí and Monacelli (2005).

2.1 Households

Household utility in period t, U_t, is a function of consumption, C_t, and labour supplied, N_t. We assume that C_t is a composite index of both domestic, $C_{H,t}$, and foreign consumption bundles, $C_{F,t}$:

$$C_t = \left[(1-\alpha)^{\frac{1}{\eta}} (C_{H,t})^{\frac{\eta-1}{\eta}} + \alpha^{\frac{1}{\eta}} (C_{F,t})^{\frac{\eta-1}{\eta}}\right]^{\frac{\eta}{\eta-1}}, \tag{1}$$

where $1-\alpha$ is the share of domestic goods in the consumption index (which also parameterizes home bias in consumption), and η is the elasticity of substitution between domestic and foreign bundles, which dictates the curvature of indifference curves between home and foreign composite goods. Unlike many papers in the welfare literature, we do not restrict the substitution elasticity between domestic and foreign goods to be unity. For Australia, Canada and New Zealand, Justiniano and Preston (2010b) estimate η to be substantially below unity.

Domestic and foreign consumption bundles are Dixit-Stiglitz aggregates of domestic and foreign goods respectively:

$$C_{H,t} = \left(\int_0^1 C_{H,t}(i)^{\frac{\tau-1}{\tau}} di \right)^{\frac{\tau}{\tau-1}} ;\text{and} \quad C_{F,t} = \left(\int_0^1 C_{F,t}(j)^{\frac{\tau-1}{\tau}} dj \right)^{\frac{\tau}{\tau-1}} \tag{2}$$

where $i \in [0,1]$ and $j \in [0,1]$ are the domestic and foreign good varieties, respectively, and τ is the elasticity of substitution between two varieties of goods when they are produced in the same country.

The demand for each variety of domestic and foreign goods is determined by finding the optimal mixture of goods for a given level of expenditure. The demand functions are:

$$C_{H,t}(i) = \left(\frac{P_{H,t}(i)}{P_{H,t}} \right)^{-\tau} C_{H,t}; \quad C_{F,t}(j) = \left(\frac{P_{F,t}(j)}{P_{F,t}} \right)^{-\tau} C_{F.t}, \tag{3}$$

for all $i, j \in [0,1]$, where

$$P_{H,t} = \left(\int_0^1 P_{H,t}(i)^{1-\tau} di \right)^{\frac{1}{1-\tau}} ; \quad P_{F,t} = \left(\int_0^1 P_{F,t}(j)^{1-\tau} dj \right)^{\frac{1}{1-\tau}} , \tag{4}$$

are the price indices for domestically produced and imported goods, respectively. Given the demand for each variety of domestic and foreign good, the optimal allocation of consumption across domestic and foreign good bundles is:

$$C_{H,t} = (1-\alpha) \left(\frac{P_{H,t}}{P_t} \right)^{-\eta} C_t; \quad C_{F,t} = \alpha \left(\frac{P_{F,t}}{P_t} \right)^{-\eta} C_t, \tag{5}$$

where

$$P_t = \left[(1-\alpha)(P_{H,t})^{1-\eta} + \alpha (P_{F,t})^{1-\eta} \right]^{\frac{1}{1-\eta}} \tag{6}$$

is the consumer price index (CPI).

We assume that the one period utility function is additively separable in the utility from consumption and the dis-utility of labour supply, such that:

$$U_t = u(C_t) - v(N_t). \tag{7}$$

And we specialize this period utility function further to be:

$$U_t = \frac{(C_t)^{1-1/\sigma}}{1-1/\sigma} - \frac{N_t^{1+1/\varphi}}{1+1/\varphi} \tag{8}$$

where σ is the elasticity of intertemporal substitution, and φ is the elasticity of labour supply.

In contrast to many empirical models, we assume that there is no habit persistence in consumption. Amato and Laubach (2004) show that the volatility of target variables is increasing in habit persistence, but Woodford (2011) shows that, while habit formation affects the feasible optimal policy frontier, it does not affect the tradeoff between policy targets. As Justiniano and Preston (2010b) estimate that habit persistence is small for New Zealand, we abstract from it in our estimation and welfare calculations.

The household maximizes utility subject to a sequence of flow budget constraints:

$$P_t C_t + E_t\left(\Lambda_{t,t+1} B_{t+1}\right) = B_t + W_t N_t + \Pi_{H,t} + \Pi_{F,t} + T_t, \tag{9}$$

where P_t is the domestic CPI (as defined earlier); B_{t+1} is the nominal payoff in period $t+1$ of a portfolio of contingent securities held at the end of period t; and $\Lambda_{t,t+1}$ is a stochastic discount factor for nominal payoffs expressed in domestic currency (discussed later). W_t is the nominal wage; T_t is lump-sum taxes and transfers; and $\Pi_{H,t}$ and $\Pi_{F,t}$ are profits earned from shares held in domestic good producers and foreign good retailers.

At time t the household faces the following intertemporal maximization problem, subject to (9):

$$\max_{C_s, N_s} E_t \left\{ \sum_{s=t}^{\infty} \beta^{s-t} \left[\frac{(C_s)^{1-1/\sigma}}{1-1/\sigma} - \frac{N_s^{1+1/\varphi}}{1+1/\varphi} \right] \right\}, \tag{10}$$

where E_t is the expectation operator and β is a discount factor.

The first order conditions for the household's maximization problem with respect to C_t and N_t are, respectively

$$\lambda_t = (C_t)^{-1/\sigma} \tag{11}$$

$$\lambda_t = N_t^{1/\varphi} \frac{P_t}{W_t} \tag{12}$$

where, in a slight abuse of notation, λ_t is the *real* shadow value of relaxing the flow budget constraint by one consumption unit at time t.

Given complete securities markets, the stochastic discount factor in the flow budget constraint, (9), is related to the consumer's optimization problem and can be represented as:

$$\Lambda_{t,s} = \beta^{s-t} \frac{u'(C_s) P_t}{u'(C_t) P_s} = \beta^{s-t} \frac{\lambda_s P_t}{\lambda_t P_s} \tag{13}$$

Where $u'(C_t)$ is the marginal utility of consumption at time t. In other words, the marginal utility from spending \$1 at time t should equal the marginal utility that could be obtained by delaying consumption and spending it at any other time period in any other state of the world (appropriately modified by the probability of that state).

For foreign consumers there is an analogous condition to (13), modified by the exchange rate:

$$\Lambda_{t,s} = \beta^{s-t} \frac{\mathcal{E}_t P_t^* u'(C_s^*)}{\mathcal{E}_s P_s^* u'(C_t^*)} = \beta^{s-t} \frac{\lambda_s^* \mathcal{E}_t P_t}{\lambda_t^* \mathcal{E}_s P_s} \tag{14}$$

where $\mathcal{E}_t$ is the nominal exchange rate expressed as domestic currency units per foreign unit. (Later we will specify $e_t \equiv \log(\mathcal{E}_t)$). Notationally, we make use of the fact that the utility functions of domestic and foreign consumers share the same functional form.

As in Galí and Monacelli (2005) and Monacelli (2005), markets are complete, implying a constant, proportional international risk sharing relationship between domestic and foreign economies' consumption. Equations (13) and (14) can be combined, implying:

$$\frac{\lambda_s P_t}{\lambda_t P_s} = \frac{\lambda_s^* \mathcal{E}_t P_t^*}{\lambda_t^* \mathcal{E}_s P_s^*} \Rightarrow$$
$$\frac{\lambda_s}{\lambda_t} = \frac{\lambda_s^* Q_t}{\lambda_t^* Q_s} \tag{15}$$

where $Q_t = \mathcal{E}_t P_t^* / P_t$ is the real exchange rate.

The degree of risk-sharing implied by complete markets is of course inconsistent with the data (Chari et al., 2002). Although we do not provide a microfounded rationale for its presence, we model the discrepancy in risk-sharing across countries by introducing a distortion ζ_t, which can be thought of as affecting the exchange rate at which consumers transact. As in Kirsanova et al. (2006), this distortion modifies the relationship between the domestic and foreign marginal rates of substitution as follows:

$$\frac{\lambda_s}{\lambda_t} = \frac{\lambda_s^* Q_t \zeta_s}{\lambda_t^* Q_s \zeta_t} \tag{16}$$

Following Galí and Monacelli (2005), and given our isoelastic utility function, home consumption is related to world consumption and the real exchange rate via

$$u'(C_t^*)\zeta_t = \vartheta Q_t u'(C_t) \Rightarrow C_t = \vartheta^\sigma Q_t^\sigma C_t^* \zeta_t \tag{17}$$

Here, ϑ is a constant that depends on intial conditions, as discussed in Galí and Monacelli (2005).[9]

The expected value of the stochastic discount factor in equation (13) is inversely related to the gross, risk-free (domestic) nominal interest rate, i_t:

$$\frac{1}{1+i_t} = E_t \Lambda_{t,t+1} = E_t \beta^{s-t} \frac{\lambda_s P_t}{\lambda_t P_s} \tag{18}$$

An analogous expression to (18) exists for nominal returns expressed in foreign currency, and using (16) we can then derive the following interest parity condition:

$$E_t \left\{ \frac{\lambda_{t+1} P_t}{\lambda_t P_{t+1}} \left[(1+i_t) - (1+i_t^*) \frac{\mathcal{E}_{t+1}\zeta_{t+1}}{\mathcal{E}_t \zeta_t} \right] \right\} = 0 \tag{19}$$

where i_t^* is the nominal risk free rate in foreign currency terms.

2.2 Domestic Production

Domestic firms produce differentiated goods using labour and subject to an AR(1) technology process, $\varepsilon_{a,t}$. The production function is given by

$$Y_t(i) = \varepsilon_{a,t} N_t(i), \tag{20}$$

[9] Note that the distortion ζ_t should be expressed as $\zeta_t^{-\frac{1}{\sigma}}$. However, following Kirsanova et al. (2006), we ignore the implicit rescaling.

where $i \in [0,1]$ indicates the firm producing good variety i. Real marginal cost is common to all firms and is expressed as

$$MC_t = \frac{W_t}{P_{H,t}\varepsilon_{a,t}}. \tag{21}$$

2.3 Price-setting behaviour: Domestic firms

Monopolistically competitive domestic firms produce differentiated, intermediate goods. These firms set prices in a staggered fashion (i.e. Calvo price-setting) allowing for indexation to the previous period's domestic goods price inflation. In a given period, a fraction $1-\theta_H$ of firms can set prices optimally, while a fraction θ_H of firms adjust prices according to the price-indexation rule

$$P_{H,t}(i) = P_{H,t-1}(i)\pi_{H,t-1}^{\delta_H}, \tag{22}$$

where δ_H is the degree of indexation to past inflation, and $\pi_{H,t}$ is the domestic goods inflation rate. Aggregate prices evolve according to

$$P_{H,t} = \left[(1-\theta_H)P'_{H,t}(i)^{1-\eta} + \theta_H\left(P_{H,t-1}\pi_{H,t-1}^{\delta_H}\right)^{1-\eta}\right]^{\frac{1}{1-\eta}}. \tag{23}$$

Because all firms that reset prices in a given period face the same decision problem, the optimal reset price is the same for all firms, $P'_{H,t}(i) = P'_{H,t}$.

Firms face a demand curve for their goods:

$$Y_{H,t}(i) = \left(\frac{P_{H,t}(i)}{P_{H,t}}\right)^{-\eta} Y_{H,t}. \tag{24}$$

Firms that can set prices in a given period maximize the present value of expected profits,

$$\max_{P'_{H,t}(i)} E_t\left\{\sum_{k=0}^{\infty}(\theta_H)^k\frac{\lambda_{t+k}}{\lambda_t}Y_{H,t+k}(i)\left[P'_{H,t}(i) - P_{H,t+k}MC_{t+k}\right]\right\}, \tag{25}$$

subject to (22) and (24). The first order condition can be expressed as a New Keynesian Philips curve for domestic goods

$$\pi_{H,t} - \delta_H\pi_{H,t-1} = \beta E_t(\pi_{H,t+1} - \delta_H\pi_{H,t}) + \frac{(1-\theta_H)(1-\beta\theta_H)}{\theta_H}mc_t, \tag{26}$$

where the quasi-differencing occurs because of the indexed price setting of firms that do not optimally reset their prices.

2.4 Price-setting behaviour: Retail firms

Monopolistic, domestic retail firms set prices in the same manner as domestic goods firms but import, rather than produce, their differentiated goods. Because retail firms have some market power and prices are set in a staggered fashion, foreign goods prices in the domestic market may differ from prices in the foreign market. Hence, in the short run the law of one price may not hold.

In a given period, a fraction $1 - \theta_F$ of retail firms can set prices optimally, while a fraction θ_F of firms adjust prices according to the price-indexation rule

$$P_{F,t}(j) = P_{F,t-1}(j)\pi_{F,t-1}^{\delta_F}, \tag{27}$$

where δ_F is the degree of indexation to past inflation, and $\pi_{F,t}$ is the domestic goods inflation rate. The domestic-currency price of aggregated imported goods evolve according to:

$$P_{F,t} = \left[(1-\theta_F)P'_{F,t}(j)^{1-\eta} + \theta_F \left(P_{F,t-1}\pi_{F,t-1}^{\delta_F}\right)^{1-\eta}\right]^{\frac{1}{1-\eta}}. \tag{28}$$

where $P'_{F,t(j)}$ denotes the price set when firm j resets their price at time t. Because all retail firms that reset prices in a given period face the same decision problem, the optimal reset price is the same for all firms, $P'_{F,t}(i) = P'_{F,t}$

Retail firms face a demand curve for their goods:

$$Y_{F,t}(j) = \left(\frac{P_{F,t}(j)}{P_{F,t}}\right)^{-\eta} Y_{F,t} \tag{29}$$

Firms that can set prices in a given period maximize the present value of expected profits,

$$\max_{P'_{F,t}(j)} E_t \left\{ \sum_{k=0}^{\infty} (\theta_F)^k \frac{\lambda_{t+k}}{\lambda_t} Y_{F,t+k}(j) \left[P'_{F,t}(i) - \mathcal{E}_{t+k}P^*_{F,t+k}\right] \right\}, \tag{30}$$

subject to (27) and (29). Given that the domestic economy has negligible size relative to the world economy, we assume that $P^*_{F,t} = P^*_t$. The first order condition can be expressed as a New Keynesian Philips curve for foreign goods sold in the domestic market:

$$\pi_{F,t} - \delta_F \pi_{F,t-1} = \beta E_t(\pi_{F,t+1} - \delta_F \pi_{F,t}) + \frac{(1-\theta_F)(1-\beta\theta_F)}{\theta_F}\psi_{F,t} + \varepsilon_{cp,t}, \tag{31}$$

where $\varepsilon_{cp,t}$ is a cost push shock.

2.5 Inflation, the real exchange rate, and the terms of trade

We define several identities as in Galí and Monacelli (2005). The terms of trade is given by

$$S_t = \frac{P_{F,t}}{P_{H,t}}, \tag{32}$$

which can be expressed in log-linear form as $s_t = p_{F,t} - p_{H,t}$. It will be useful to rewrite (32) and the CPI equation as:

$$\frac{P_{H,t}}{P_t} = \left[(1-\alpha) + \alpha S_t^{1-\eta}\right]^{\frac{-1}{1-\eta}}. \tag{33}$$

In a first order log-linear approximation, (33) can be expressed as

$$p_t = p_{H,t} + \alpha s_t \tag{34}$$

where, in steady state, $\overline{S} = 1$, $P_H = P_F$, and purchasing power parity (PPP) holds. CPI inflation is:

$$\pi_t = p_t - p_{t-1}, \tag{35}$$

Given (34), CPI inflation can be expressed as a relationship between domestic inflation, $\pi_{H,t}$, and the terms of trade, s_t:

$$\begin{aligned} \pi_t &= \pi_{H,t} + \alpha \Delta s_t \\ &= (1-\alpha)\pi_{H,t} + \alpha \pi_{F,t} \end{aligned} \tag{36}$$

Thus, CPI inflation is a composite of domestic and foreign good inflation. Moreover, the difference between CPI and domestic inflation is proportionate to the change in the terms of trade and explicitly depends on the degree of openness α.

Because retail firms hold a degree of market power, the domestic price of imported goods is not always the domestic currency equivalent of the foreign price of those goods. That is, the law of one price does not always hold because exchange rate movements only pass-through imperfectly to domestic import prices, i.e. $p_{F,t} \neq e_t + p_t^*$.

The real exchange rate is

$$Q_t = \frac{\mathcal{E}_t P_t^*}{P_t}. \tag{37}$$

Kirsanova et al. (2006) show that with the definition of the CPI and the terms of trade equation (34), the real exchange rate can be expressed as

$$\begin{aligned} Q_t &= \mathcal{E}_t P_t^* [(1-\alpha)(P_{H,t})^{1-\eta} + \alpha (P_{F,t})^{1-\eta}]^{-\frac{1}{1-\eta}} \\ &= \frac{\mathcal{E}_t P_t^*}{P_{F,t}} [(1-\alpha)(P_{H,t})^{1-\eta} P_{F,t}^{-(1-\eta)} + \alpha (P_{F,t})^{1-\eta} P_{F,t}^{-(1-\eta)}]^{-\frac{1}{1-\eta}} \\ &= \frac{\mathcal{E}_t P_t^*}{P_{F,t}} [(1-\alpha) S_t^{-(1-\eta)} + \alpha]^{-\frac{1}{1-\eta}} \\ &= \Psi_{F,t} [(1-\alpha) S_t^{-(1-\eta)} + \alpha]^{-\frac{1}{1-\eta}}, \end{aligned} \tag{38}$$

where $\Psi_{F,t}$ is the difference between the domestic currency price of imports and the foreign price of those goods – the law of one price gap.[10] Note that as the economy becomes more open ($\alpha \to 1$), the real exchange rate only fluctuates in response to law of one price gaps. Hence, incomplete pass-through, via law of one price gaps, is a source of real exchange rate volatility.

2.6 General equilibrium

Equilibrium in the domestic goods market occurs when:

$$Y_{H,t} = C_{H,t} + C_{H,t}^*. \tag{39}$$

Foreign demand for domestic goods is given by:

$$C_{H,t}^* = \alpha^* \left(\frac{P_{H,t}^*}{P_t^*} \right)^{-\eta} C_t^*, \tag{40}$$

[10] When $\eta = 1$ the log-linear form of the real exchange rate is an exact linear relationship $q_t = \Psi_{F,t} + (1-\alpha)s_t$, which is the relationship between the real exchange rate, terms of trade, and law of one price gap in Monacelli (2005).

where α^* is the share of domestic goods in the foreign consumption index, $P^*_{H,t}$ is the foreign price of the domestic goods bundle, and P^*_t is the foreign aggregate price level. Note that the elasticity of substitution between domestic and foreign goods in the foreign economy, η, is assumed to be the same as in the domestic economy.[11]

2.7 Monetary policy rule

The monetary authority is the final actor in our model. We assume that the monetary authority implements the following augmented Taylor rule:

$$i_t = \rho_i i_{t-1} + (1-\rho_i)\left(\psi_\pi \pi_t + \psi_y y_t + \psi_e \Delta e_t + \psi_{\Delta y} \Delta y_t\right) + \varepsilon_{M,t} \qquad (41)$$

Policy parameters are denoted $\theta_p = [\rho_i, \psi_\pi, \psi_y, \psi_{\delta y}, \psi_e]'$. As we model a small open economy, we assume that there is no strategic interaction between foreign and domestic monetary authorities.

Note that the policy rule only responds to the lagged interest rate, CPI inflation, output gaps, nominal exchange rate depreciations, and the change in the output gap. It is thus unlikely to reflect the form of an optimal rule. Galí and Monacelli (2005), for example, show that because the welfare distortion caused by price stickiness manifests itself in domestic price inflation, optimal policy responds to domestic prices rather than the CPI. Nevertheless, here we are interested in the estimated and optimal parameterizations of a familiar class of Taylor rules that have previously been explored in the literature.[12]

2.8 Stochastic shocks

There are six exogenous disturbances that enter our model: the technology shock; a disturbance to interest rates (a monetary policy shock), the cost-push shock affecting the price of foreign goods, the disturbance to risk sharing, a world output demand shock, and a world price shock (which affects the real exchange rate). The risk-sharing shock and the monetary policy shock are both modelled as independent, identically distributed (IID) processes, while the other four shocks are modelled as autoregressions of order one, AR(1). The estimated risk-sharing shock appear to be autocorrelated, and hence inconsistent with the IID assumption, but computational difficulties made it impractical to estimate an autocorrelaion coefficient for this shock.

2.9 The second order approximation to the model

We summarize the demand side of the model in two equations determining output and consumption. We then take a second order approximation to these equations in order to substitute them into the second order approximation to the utility function, following the method of Kirsanova et al. (2006).

[11] Justiniano and Preston (2010b) report that allowing for differences in η across economies does not greatly affect the estimation of the model.

[12] Justiniano and Preston (2010b) and Kam et al. (2009) describe the same rule as (41), while Lubik and Schorfheide (2006) employ the restriction $\psi_{\Delta y} = 0$.

Substituting the demand functions (5) and (40) into the equilibrium condition, (39), yields

$$\begin{aligned} Y_{H,t} &= (1-\alpha)\left(\frac{P_{H,t}}{P_t}\right)^{-\eta} C_t + \alpha^* \left(\frac{P^*_{H,t}}{P^*_t}\right)^{-\eta} C^*_t \\ &= (1-\alpha)\left(\frac{P_{H,t}}{P_t}\right)^{-\eta} C_t + \alpha^* \left(\frac{\mathcal{E}_t P^*_{H,t}}{P_t}\frac{P_t}{\mathcal{E}_t P^*_t}\right)^{-\eta} C^*_t \\ &= (1-\alpha)\left(\frac{P_{H,t}}{P_t}\right)^{-\eta} C_t + \alpha^* \left(\frac{P_{H,t}}{P_t}\right)^{-\eta} Q^\eta_t C^*_t, \end{aligned} \tag{42}$$

where the final equality holds if the law of one price holds in the foreign economy, i.e. $\mathcal{E}_t P^*_{H,t} = P_{H,t}$. Now, using (33), we have

$$Y_{H,t} = \left[(1-\alpha) + \alpha S_t^{1-\eta}\right]^{\frac{\eta}{1-\eta}} \left[(1-\alpha)C_t + \alpha^* Q^\eta_t C^*_t\right]. \tag{43}$$

Substituting the risk sharing relationship (17) into (42) gives

$$Y_{H,t} = \left(\frac{P_{H,t}}{P_t}\right)^{-\eta} \left[(1-\alpha)\vartheta^\sigma Q^\sigma_t C^*_t \zeta_t + \alpha^* Q^\eta_t C^*_t\right]. \tag{44}$$

With $\vartheta = 1$, $\alpha = \alpha^*$, and using the two real exchange rate equations $Q_t = \frac{\mathcal{E}_t P^*_t}{P_t}$ and (38), we have

$$Y_{H,t} = S^\eta_t C^*_t \left[(1-\alpha)\psi^{\sigma-\eta}_{F,t} \left[(1-\alpha)S^{-(1-\eta)} + \alpha\right]^{\frac{-(\sigma-\eta)}{1-\eta}} \zeta_t + \alpha\right]. \tag{45}$$

Also using the above assumptions, we can write the international risk sharing equation (17) as

$$C_t = C^*_t \psi^\sigma_{F,t} \left[(1-\alpha)S_t^{-(1-\eta)} + \alpha\right]^{-\frac{\sigma}{1-\eta}} \zeta_t. \tag{46}$$

Together the output and consumption equations, (45) and (46), form the system of equations from which we derive the second order approximation to the welfare function.

Note, our first-order expansions make use of the log-deviation of a variable X_t from its steady state X. Second order expansions make use of the fact that $\frac{X_t}{X} = 1 + \hat{X}_t + \frac{1}{2}\hat{X}^2_t$, where $\hat{X}_t \equiv \log(X_t) - \log(X)$.

The first order expansions of equations (45) and (46) are:

$$\begin{aligned} \hat{Y}_t =& \hat{C}^*_t + (1-\alpha)(\sigma-1)\hat{\psi}_{F,t} \\ &+ (\alpha^2\sigma - \alpha^2\eta - 2\alpha\sigma + 2\alpha\eta + \sigma)\hat{S}_t + (1-\alpha)\hat{\zeta}_t, \end{aligned} \tag{47}$$

and

$$\hat{C}_t = \hat{C}^*_t + \sigma\hat{\psi}_{F,t} + \sigma(1-\alpha)\hat{S}_t + \hat{\zeta}_t. \tag{48}$$

The second order expansions are:

$$\begin{aligned}\hat{Y}_t =& \hat{C}_t^* + (\alpha^2\sigma - \alpha^2\eta - 2\alpha\sigma + 2\alpha\eta + \sigma)\hat{S}_t + (1-\alpha)\hat{\zeta}_t \\ &+ \frac{1}{2}(\eta^2 + (1-\alpha)^2(\sigma-\eta)(3\eta - 1 + (1-\alpha)(\sigma - 2\eta + 1)))\hat{S}_t^2 \\ &- \frac{1}{2}\hat{Y}_t^2 + (\alpha\sigma - \alpha\eta - \sigma)(\alpha - 1)\hat{S}_t\hat{\zeta}_t \\ &+ (\alpha^2\sigma - \alpha^2\eta - 2\alpha\sigma + 2\alpha\eta + \sigma)\hat{S}_t\hat{C}_t^* - \frac{1}{2}(\alpha-1)\hat{\zeta}_t^2 + (1-\alpha)\hat{\zeta}_t\hat{C}_t^* + \frac{1}{2}\hat{C}_t^{*2} \\ &+ (1-\alpha)(\sigma-\eta)\hat{\psi}_{F,t} + \frac{1}{2}(\sigma-\eta)^2(1-\alpha)\hat{\psi}_{F,t}^2 + (\sigma-\eta)(1-\alpha)(\sigma - \alpha\sigma + \alpha\eta)\hat{\psi}_{F,t}\hat{S}_t \\ &+ (\sigma-\eta)(1-\alpha)\hat{C}_t^*\hat{\psi}_{F,t} + (\sigma-\eta)(1-\alpha)\hat{\psi}_{F,t}\hat{\zeta}_t + \mathcal{O}(3)\end{aligned} \tag{49}$$

and

$$\begin{aligned}\hat{C}_t =& -\frac{1}{2}\hat{C}_t^2 + \hat{C}_t^* + \frac{1}{2}\hat{C}_t^{*2} + \sigma\hat{\psi}_{F,t} + \frac{1}{2}\sigma^2\hat{\psi}_{F,t}^2 + \sigma(1-\alpha)\hat{S}_t \\ &+ \frac{1}{2}\sigma(1-\alpha)(\sigma(1-\alpha) - \alpha(1-\eta))\hat{S}_t^2 \\ &+ \hat{\zeta}_t + \frac{1}{2}\hat{\zeta}_t^2 + \sigma\hat{\psi}_{F,t}\hat{C}_t^* + \sigma(1-\alpha)\hat{S}_t\hat{C}_t^* + \hat{\zeta}_t\hat{C}_t^* \\ &+ \sigma^2(1-\alpha)\hat{S}_t\hat{\psi}_{F,t} + \sigma\hat{\zeta}_t\hat{\psi}_{F,t} + \sigma(1-\alpha)\hat{\zeta}_t\hat{S}_t + \mathcal{O}(3)\end{aligned} \tag{50}$$

where $\mathcal{O}(3)$ denotes terms of third order or higher.

2.10 Flexible price and efficient equilibria

Shocks to international risk sharing do not disappear in the flexible price equilibrium. As a result, the natural rates of output, consumption, and the terms of trade are not efficient; they are buffeted by distortionary shocks to the exchange rate. We distinguish between the flexible price equilibrium and the efficient equilibrium by denoting the flexible price ('natural') variables $\hat{X}^n$ for any variable X, and efficient equilibrium $\hat{X}^e$. Our welfare analysis proceeds using the deviation of variables from their efficient rates: $\hat{X} - \hat{X}^e$.[13]

In the flexible price equilibrium, the marginal rate of substitution between consumption and labour is equal to the real wage, which is equal to the real price of the domestic good (i.e. the price of the domestic good deflated by the CPI). In order to account for the monopolistic distortion to steady state and thus flexible price equilibrium, we include an employment subsidy μ^w to offset the producer's price markup μ. The relationship can be written

$$\frac{v'(N_t^n)}{u'(C_t^n)} = \frac{W_t}{P_t} = \varepsilon_{a,t}\frac{\mu^w}{\mu}\frac{P_{H,t}}{P_t}, \tag{51}$$

where n superscripts denote flexible price, or natural rate, variables. Substituting into this the first order conditions of the utility function and the production function yields

$$Y_t^{n\frac{1}{\varphi}}\varepsilon_{a,t}^{-\frac{1}{\varphi}}C_t^{n\frac{1}{\sigma}} = \frac{W_t}{P_t} = \varepsilon_{a,t}\frac{\mu^w}{\mu}\frac{P_{H,t}}{P_t}. \tag{52}$$

[13] Kirsanova et al. (2006) note the possibility of decomposing gap variables into the form $\hat{X} - \hat{X}^e = \left(\hat{X} - \hat{X}^n\right) - \left(\hat{X}^e - \hat{X}^n\right)$, reflecting the difference between actual equilibrium deviations from the natural rate and natural rate deviations from efficient equilibrium.

If we now substitute the terms of trade, $S_t = \frac{P_{F,t}}{P_{H,t}}$, into the CPI equation, $P_t = \left[(1-\alpha)P_{H,t}^{1-\eta} + \alpha P_{F,t}^{1-\eta}\right]^{\frac{1}{1-\eta}}$, and rearrange, we have

$$\frac{P_t}{P_{H,t}} = \left[(1-\alpha) + \alpha S_t^{n\ 1-\eta}\right]^{\frac{1}{1-\eta}} \tag{53}$$

Substituting this into (52) and log-linearizing, we can express the relationship between the natural rates and the technology shock as

$$\sigma \hat{Y}_t^n + \psi \hat{C}_t^n + \alpha\varphi\sigma \hat{S}_t^n = \sigma(1+\varphi)\varepsilon_{a,t} \tag{54}$$

With the monopolistic distortion dealt with, we can now derive expressions for the efficient levels of variables from the first order approximations to the model. The efficient equilibrium does not feature international risk sharing shocks, and because prices are flexible, law of one price gaps are always zero, i.e. $\hat{\psi}_{F,t}^n = \hat{\psi}_{F,t}^e = 0$. The efficient rates of output and consumption are

$$\hat{Y}_t^e = \hat{C}_t^* + (\alpha^2\sigma - \alpha^2\eta - 2\alpha\sigma + 2\alpha\eta + \sigma)\hat{S}_t^e \tag{55}$$

$$\hat{C}_t^e = \hat{C}_t^* + \sigma(1-\alpha)\hat{S}_t^e. \tag{56}$$

Solving the system of equations consisting of (54), (55), and (56) yields

$$\hat{C}_t^e = \frac{\sigma^2(1-\alpha)(1+\varphi)}{\sigma(2\alpha\eta + \sigma + \varphi - 2\alpha\sigma - \alpha^2\eta + \alpha^2\sigma)}\varepsilon_{a,t} + \frac{\alpha(\varphi - \sigma - \alpha\eta + \alpha\sigma + 2\eta)}{(2\alpha\eta + \sigma + \varphi - 2\alpha\sigma - \alpha^2\eta + \alpha^2\sigma)}\hat{C}_t^* \tag{57}$$

$$\hat{S}_t^e = \frac{\sigma(1+\varphi)}{\sigma(2\alpha\eta + \sigma + \varphi - 2\alpha\sigma - \alpha^2\eta + \alpha^2\sigma)}\varepsilon_{a,t} - \frac{(\sigma+\varphi)}{\sigma(2\alpha\eta + \sigma + \varphi - 2\alpha\sigma - \alpha^2\eta + \alpha^2\sigma)}\hat{C}_t^* \tag{58}$$

$$\hat{Y}_t^e = \frac{\sigma(1+\varphi)(\alpha^2\sigma - \alpha^2\eta - 2\alpha\sigma + 2\alpha\eta + \sigma)}{\sigma(2\alpha\eta + \sigma + \varphi - 2\alpha\sigma - \alpha^2\eta + \alpha^2\sigma)}\varepsilon_{a,t} + \frac{\alpha\varphi(\eta - \sigma)(\alpha - 2)}{\sigma(2\alpha\eta + \sigma + \varphi - 2\alpha\sigma - \alpha^2\eta + \alpha^2\sigma)}\hat{C}_t^* \tag{59}$$

Hence, the efficient rates of consumption, the terms of trade, and output are functions of foreign demand and domestic technology shocks.

3 International frictions and imperfections

In the introduction of the paper we noted that in the presence of the terms of trade externality, home bias, or imperfect risk-sharing, monetary policy has a reason to stabilise the exchange rate. In this section we briefly outline some of the relevant mechanisms.

In prototypical DSGE models prices induce private agents to allocate consumption and labour across time, and to allocate consumption across varieties of goods. Price stickiness, in conjunction with shocks, means that some prices will be misaligned relative to optimal marginal rates of substitution and transformation. As Galí (2008) discusses, the welfare costs of such shocks and such rigidities depends on the magnitude of shocks, on how much prices should move in response to the shocks, and on how responsive the allocation of resources is to changes in prices. The former depends on the degree of price stickiness and other rigidities, while the latter depends on the curvature of utility functions and production functions, such as the intertemporal and intratemporal substitution elasticities, the degree of home bias, and on the curvature of the production function (which also depends on the labour supply elasticity). When the intertemporal and intratemporal substitution elasticities are unitary many of the allocative effects of shocks to prices wash out because the income and substitution effects of the change in prices just offset each other.

? notes that when foreign and domestic goods are imperfect substitutes (in the notation below $\eta < \infty$) there is an incentive to impose a tax or tariff on exports to reduce supply and hence increase their price, and if import supply is imperfectly inelastic there is an incentive to impose a tax or tariff on imported goods. The optimal increase in price depends on the substitutability of domestic and foreign goods. In the context of open economy macro models with interest rate rules, the tariff or tax is implemented by altering domestic interest rates and hence the exchange rate.

De Paoli (2009a) takes a second order approximation of a non-linear small open economy model and shows that a less volatile real exchange rate is associated with an appreciated exchange rate (i.e. an appreciated terms of trade). Policies that reduce real exchange rate volatility thus enables policy-makers to take advantage of the terms of trade externality. The type of policy that will stabilise the real exchange rate depends on the underlying structural shocks that are causing volatility. As discussed in De Paoli (2009a), greater substitutability between domestic and foreign goods means that, in response to a terms of trade improvement, consumption of foreign goods can be substituted for domestic consumption, concomitant with reduction in work effort, which overall improves welfare.

Conditional on productivity shocks being the main driver of fluctuations, a fixed exchange rate will be optimal when, as per De Paoli's analysis, the intratemporal substitution elasticity is high. However, Galí (2008, p. 169) suggests the required degree of substitutability is implausibly high, and that stabilising domestic prices remains broadly optimal.

4 The welfare function

The utility function specified in equation (10) illustrates that welfare losses will only occur when either consumption or labour effort are distorted from their efficient levels. We derive a second order approximation to the utility function to understand how shocks perturb welfare. Following Galí (2008), the second order Taylor expansion of the utility function is:

$$
\begin{aligned}
U_t - U =& Cu'(C)\left[\widehat{C}_t + \frac{1}{2}(1-\frac{1}{\sigma})\widehat{C}_t^2\right] \\
& - Nv'(N)\left[\widehat{Y}_t + \frac{1}{2}(1+\frac{1}{\varphi})\widehat{Y}_t^2 - (1+\frac{1}{\varphi})\widehat{Y}_t\hat{\varepsilon}_{a,t} + \frac{\tau}{2}\mathrm{Var}_j\widehat{p}_{H,t}(j)\right] \\
& + \text{t.i.p.} + \mathcal{O}(3)
\end{aligned}
\tag{60}
$$

where t.i.p. represents terms independent of policy (note that this will later include the shock terms) and, as before, $\mathcal{O}(3)$ denotes terms of third order or higher.

We can now substitute the second order expansions of output and consumption, (49) and (50), into the second order expansion of the utility function and, assuming the presence of an optimal tax to eliminate the monopolistic distortion in the production of domestic goods (see Woodford 2011), we can represent social welfare as

$$\begin{aligned}\mathcal{W}_t =& A_1\left(\hat{Y}_t - \hat{Y}_t^e\right)^2 + A_2(\hat{\psi}_{F,t} - \hat{\psi}_{F,t}) + A_3(\hat{Y}_t - \hat{Y}_t^e)\hat{Y}_t^* \\ &+ A_4(\hat{Y}_t - \hat{Y}_t^e)\hat{\varepsilon}_{a,t} + A_5\left(\hat{Y}_t - \hat{Y}_t^e\right)\hat{\zeta}_t + A_6(\hat{Y}_t - \hat{Y}_t^e)(\hat{\psi}_{F,t} - \hat{\psi}_{F,t}^e) \\ &+ A_7(\hat{\psi}_{F,t} - \hat{\psi}_{F,t}^e)^2 + A_8(\hat{\psi}_{F,t} - \hat{\psi}_{F,t}^e)\hat{Y}_t^* + A_9(\hat{\psi}_{F,t} - \hat{\psi}_{F,t}^e)\hat{\varepsilon}_{a,t} \\ &+ A_{10}(\hat{\psi}_{F,t} - \hat{\psi}_{F,t}^e)\hat{\zeta}_t + A_{11}\mathrm{Var}_j\widehat{p}_{H,t}(j) + \text{t.i.p.} + \mathcal{O}(3)\end{aligned} \tag{61}$$

where A_k, for $k = 1, \ldots, 11$, are functions of the structural parameters, as shown in the appendix.

Finally, the policymaker's loss function is given by

$$\begin{aligned}\mathcal{L}_0 =& E_0\sum_{t=0}^{\infty}\beta^t\mathcal{W}_t = A_1\mathrm{Var}\left(\widetilde{Y}_t\right) + A_3\mathrm{Cov}\left(\widetilde{Y}_t, \hat{Y}_t^*\right) + A_4\mathrm{Cov}\left(\widetilde{Y}_t, \hat{\varepsilon}_{a,t}\right) \\ &+ A_5\mathrm{Cov}\left(\widetilde{Y}_t, \hat{\zeta}_t\right) + A_6\mathrm{Cov}\left(\widetilde{Y}_t, \widetilde{\psi}_{F,t}^e\right) + A_7\mathrm{Var}\left(\widetilde{\psi}_{F,t}^e\right) + A_8\mathrm{Cov}\left(\widetilde{\psi}_{F,t}^e, \hat{Y}_t^*\right) \\ &+ A_9\mathrm{Cov}\left(\widetilde{\psi}_{F,t}^e, \hat{\varepsilon}_{a,t}\right) + A_{10}\mathrm{Cov}\left(\widetilde{\psi}_{F,t}^e, \hat{\zeta}_t\right) + A_{11}\mathrm{Var}\left(\pi_{H,t} - \delta_H\pi_{H,t-1}\right),\end{aligned} \tag{62}$$

where $\left(\hat{Y}_t - \hat{Y}_t^e\right) = \widetilde{Y}_t$ is the deviations of the output gap from the efficient output gap, and $(\hat{\psi}_{F,t} - \hat{\psi}_{F,t}^e) = \widetilde{\psi}_{F,t}^e$ is the deviations of the law of one price gap from the efficient law of one price gap (i.e. zero). Note that the term $A_2(\hat{\psi}_{F,t} - \hat{\psi}_{F,t})$ in the welfare function drops out from the loss function as $E_0(\hat{\psi}_{F,t} - \hat{\psi}_{F,t}) = 0$.

Here, we have represented the variables in the welfare function in terms of deviations from the efficient equilibrium levels of those variables. As Kirsanova et al. (2006) note, in the presence of distortions other than sticky nominal domestic goods prices, the flexible price equilibrium will not necessarily be efficient. In the presence of international risk sharing shocks the optimal path of consumption diverges from the optimal path of output due to fluctuations in the terms of trade. Closing the output gap does not close the consumption gap, hence there is policy justification for allowing output to differ from its natural/flexible price rate.

Galí and Monacelli (2005) consider a restricted case where the small open economy does not experience law of one price gaps, and the structure of the economy is such that $\sigma = \eta = 1$. When this is the case, the loss function reduces to

$$\mathcal{L}_0 = A_1'\mathrm{Var}\left(\widetilde{Y}_t\right) + A_4'\mathrm{Cov}\left(\widetilde{Y}_t, \hat{\varepsilon}_{a,t}\right) + A_{11}'\mathrm{Var}\left(\pi_{H,t} - \delta_H\pi_{H,t-1}\right). \tag{63}$$

When the multiplicative term in the technology shock is bundled into the output gap term (see Galí 2008), the welfare function illustrates the optimal monetary policy isomorphism with the closed economy described in Galí and Monacelli (2005) and Clarida et al. (2001).[14] Conversely, the assumption that $\sigma = \eta = 1$

[14] The sole difference being that with indexation of domestic prices, inflation losses manifest themselves as a quasi-difference term, $\mathrm{Var}\left(\pi_{H,t} - \delta_H\pi_{H,t-1}\right)$, rather than the the rate of inflation itself, $\mathrm{Var}\left(\pi_{H,t}\right)$, as in Galí and Monacelli (2005) and Clarida et al. (2001).

in a small open economy setting with distortions other than domestic price stickiness obscures the multi-faceted nature of social welfare. Under less restrictive assumptions, in addition to the output gap and domestic inflation, the policy maker is concerned with losses in law of one price gaps, and losses induced by covariances between the shocks in the model and the output and law of one price gaps.

In a small open economy with $\sigma = 1$, consumption and the terms of trade move in proportion to the output gap as can be seen in (47) and (48).This proportionality means that welfare losses do not arise due to movements in consumption or the terms of trade independent of losses in the output gap. In this world, policy makers thus need only concern themselves with output and inflation.

Kirsanova et al. (2006) explain that when $\sigma \neq 1$ and there are international risk sharing shocks, this proportionality is broken because output gaps and consumption and terms of trade gaps can move independently. For example, a positive, one unit international risk sharing shock could lead to a one unit rise in consumption, but only a $(1 - \alpha)$ rise in output (see equation (48) and (47)). International risk sharing shocks introduce a role for the terms of trade, and thus the exchange rate, in welfare and policy setting.

Introducing sticky retail import prices reveals an additional role for the exchange rate via law of one price gaps. Kirsanova et al. (2006) discuss the finding that despite the introduction of the role of the exchange rate as a result of the inclusion of IRS shocks, domestic output inflation, rather than CPI inflation, is the appropriate focus of policy. Similarly, we find that despite the introduction of sticky retail import good prices, and thus an import goods Philips curve, there is no role for imported goods price inflation and so no role for CPI inflation (insofar as the CPI is a weighted average of domestic and foreign goods price inflations). The reason for the difference in treatment of domestic goods price inflation and foreign goods price inflation is as follows. Dispersion in the prices of domestic goods leads to differences across firms in demand, labour, and thus the marginal disutility of labour supplied. Retailers of imported goods do not use labour as an input in production, so dispersion in prices does not affect the marginal disutility of labour. In fact, price dispersion of imported goods does not matter at all for welfare. What matters is the speed with which the aggregate price level of imported goods adjusts to changes in the foreign price of imports and the exchange rate. The slower is this adjustment, the larger and more persistent law of one price gaps can be.

5 Estimation

5.1 Data

Our estimation uses quarterly data for New Zealand's output, inflation, interest rates, terms of trade, and the real exchange rate. A foreign price level is constructed using trade (and GDP-weighted) foreign data, as discussed in more depth below. The sample period is 1990:1 to 2012:4, and all variables (except interest rates) are measured in logs and are demeaned. All interest rates and log differences are expressed in quarterly decimal terms, i.e. an interest rate of 5 percent per annum is recorded as 5/400 in quarterly period terms.

Domestic output is seasonally adjusted, real, production-based GDP per capita. This GDP per capita series is detrended with a stiff HP filter using $\lambda = 10,000$, which is similar to, but more flexible than, the linear filter employed in Justiniano and Preston (2010b). The HP filter parameterized in this way yields an output gap that has similar properties to a output gap measure actively used by the Reserve Bank of New Zealand to forecast non-tradables inflation. Domestic inflation is computed using the log difference of the all-groups consumers price index. The domestic interest rate is the 90-day bank bill rate. The terms of trade is New Zealand's all-country terms of trade index.

Justiniano and Preston (2010b) construct data for foreign variables using US data. While the US is a reasonable summary of the 'foreign block' for Canada it is less appropriate for New Zealand and Australia. In New Zealand's case, for example, the US accounts for less than 20 percent of exports, and approximately the same proportion of imports. Liu (2006) uses an 80:20 US-Australian weighting for foreign variables reflecting the fact that much of New Zealand's trade is carried out with Australia. In order to capture more of New Zealand's trading activity, we construct data for the foreign variables using the weighting method that applies to the Reserve Bank of New Zealand's Trade Weighted Index (TWI) for the exchange rate. The TWI captures a weighted basket of New Zealand's trading partners: Australia, the US, the UK, Japan, and the euro area. The weights are determined annually by a 50:50 weighting of New Zealand's import and export merchandise trade and the GDP-share of each country.[15]

The first difference in the real exchange rate is the depreciation rate of the real TWI (i.e. the exchange rate is units of domestic currency per foreign currency, so the domestic currency depreciates if the exchange rate series increases). Foreign inflation is constructed as the CPI inflation rate of New Zealand's major trading partners weighted according to the TWI method. This foreign inflation series is the only foreign observable used in the estimation.

5.2 Parameter estimates

The priors for our Bayesian estimation are largely similar to those used by Justiniano and Preston (2010b). The model is solved and the parameters are estimated using Dynare.[16] Numerical algorithms to estimate the parameters are initialized with Justiniano and Preston's median posterior parameter estimates.[17] As the elasticity of substitution between domestic varieties is not estimated in that model, we assume that $\tau = 6$.[18]

The parameters of the model are reported in table 1. The variance decomposition specifying the main exogenous drivers of fluctuations in the endogenous

[15] More information about the TWI can be found on the Reserve Bank website: `http://www.rbnz.govt.nz/statistics/exandint/twi/index.html`.

[16] See Adjemian et al. (2011).

[17] The model was also re-parameterized to improve the condition of the Hessian of the parameter vector. The autoregressive coefficient on foreign output and the domestic Calvo coefficient were both divided by 10 and the prior distributions were rescaled accordingly.

[18] τ appears in our loss function but does not appear in the log-linearized equations of the model economy. Woodford (2011) and Kirsanova et al. (2006) similarly assume $\tau = 6$, which implies a steady state markup of 20 percent. As can be seen from the values of the welfare function parameters in the appendix, increasing τ simply increases the weight on the variance in inflation, reinforcing the results discussed in 6.2.

variables is reported in table 2.

Table 1: Prior and posterior model parameters

Parameter	Description	Prior	Mean	StdDev	Mean	(5%	95%)
α	Home bias parameter	Γ	0.290	0.050	0.092	0.075	0.106
$1/\sigma$	Inverse intertemporal substitution elasticity	Γ	1.200	0.400	1.177	0.567	1.786
$1/\varphi$	Inverse labour supply elasticity	Γ	1.500	0.750	1.490	0.320	2.639
θ_h	Calvo parameter domestic goods	β	5.000	1.000	7.151	6.538	7.779
θ_f	Calvo parameter foreign goods	β	0.500	0.100	0.063	0.047	0.076
η	Foreign/home substitution elasticity	Γ	1.500	0.750	0.272	0.114	0.422
δ_h	Price indexation domestic goods	β	0.500	0.250	0.086	0.001	0.177
δ_f	Price indexation foreign goods	β	0.500	0.250	0.724	0.434	0.999
ψ_i	Policy coef. on lag interest rate	β	0.500	0.250	0.039	0.009	0.076
ψ_π	Policy coef. on inflation	Γ	1.500	0.300	1.338	0.915	1.759
ψ_y	Policy coef. on output gap	Γ	0.250	0.130	2.035	1.961	2.092
$\psi_{\Delta e}$	Policy coef. on change in exchange rate	Γ	0.250	0.130	0.090	0.023	0.152
ρ_a	Technology persistence AR(1)	β	0.800	0.100	0.477	0.346	0.611
ρ_{cp}	Cost push shock persistence AR(1)	β	0.500	0.250	0.921	0.864	0.981
$B_{y,y}$	Foreign VAR(1) parameter	N	9.000	5.000	7.675	6.343	8.971
$B_{\pi,\pi}$	Foreign VAR(1) parameter	N	0.500	0.500	0.322	-0.234	0.925
σ_a	Std dev. of technology shock	Γ^{-1}	3.000	5.000	1.322	0.711	1.867
σ_m	Std dev. of interest rate shock	Γ^{-1}	3.000	5.000	0.409	0.401	0.419
σ_ζ	Std dev. risk-sharing shock	Γ^{-1}	3.000	5.000	0.482	0.401	0.548
σ_{cp}	Std dev. of cost push shock	Γ^{-1}	3.000	5.000	0.708	0.533	0.879
σ_{y*}	Std dev. of fx output shock	Γ^{-1}	3.000	5.000	0.412	0.401	0.427
$\sigma_{\pi*}$	Std dev. of fx inflation shock	Γ^{-1}	3.000	5.000	0.409	0.401	0.419

Note: θ_h and $B_{y,y}$ have been rescaled by a factor of 10 for numerical reasons.

The posterior analysis indicates that the degree of home bias is much greater than the prior might have suggested, and the data are quite informative about this parameter. Conversely, the data are essentially uninformative about the intertemporal substitution elasticity and the labour supply elasticity, and the posterior is thus dominated by their respective prior distributions.

The posteriors imply that the substitution elasticity between home and foreign goods, η, is very low, a result driven by the likelihood rather than the prior. As Tille (2001) notes, it may be quite reasonable to expect countries to specialize in the production of particular types of goods, in which case the intratemporal substitution elasticity would be lower than the elasticity of substitution between varieties of goods within single country. McDaniel and Balistreri (2003) and others make clear that estimated intatemporal susbstitution elasticities vary greatly, depending on both the level of industry aggregation and the econometric methods used to estimate these parameters. Nevertheless, it is clear that our estimate is at the low end of the spectrum; Feenstra et al. (2010), for example, suggest that estimates of the intratemporal elasticity between home and foreign goods are usually about unity for the United States, irrespective of the particular sector being examined. However, it is conceivable that New Zealand's unique production bundle may reduce the applicability of foreign elasticity estimates.

Estimated price setting behaviour across the home and foreign sectors is somewhat disparate: the Calvo parameter for home goods is 0.72, but only 0.06 for foreign goods, implying that most importing firms do not suffer from sticky prices. Further, the estimation implies that there is much more indexation in the foreign goods sector than in the domestic sector.

The estimates of the policy rule are quite different to those found in Justiniano and Preston (2010b) and Lubik and Schorfheide (2006). Because of computational difficulties, the parameter on the change in the output gap was set to zero. What is most notable in the estimated parameters is that there is very little interest rate smoothing, and the estimated rule implies very vigorous responses both to inflation and the output gap, with a negligible response to the change in the nominal exchange rate. However, persistence is maintained in interest rates via the persistence of the underlying arguments entering the policy rule.

Of the exogenous shock processes, the cost push shock is found to be most persistent, followed by the world demand shock. The persistence in the technology shock is low, and the persistence in the foreign inflation shock is even lower. Table 2 shows that despite its lack of persistence, the interest rate shock is the primary driver of cyclical fluctuations in output and interest rates, but is of negligible importance for the other observed variables. The international risk sharing shock and the foreign demand shock are the most important drivers of inflation, the terms of trade, and the exchang rate. Surprisingly, the technology shock plays little role in driving cyclical fluctuations in the variables reported. Out of interest, we estimated a version of the model with a (mild) calibrated degree of habit persistence, but this did not change the qualitative features of the variance decomposition.[19]

[19] We set the degree of habit persistence to $h = 0.08$, as estimated for New Zealand by Justiniano and Preston

Table 2: Variance decomposition

	ϵ_a	ϵ_m	ϵ_ζ	ϵ_{cp}	ϵ_{y*}	$\epsilon_{\pi*}$
y	0.00	0.85	0.08	0.00	0.05	0.02
	[0.00, 0.00]	[0.74, 0.94]	[0.03, 0.13]	[0.00, 0.00]	[0.02, 0.08]	[0.00, 0.04]
π	0.01	0.05	0.63	0.00	0.31	0.00
	[0.00, 0.01]	[0.04, 0.05]	[0.61, 0.65]	[0.00, 0.01]	[0.29, 0.32]	[0.00, 0.00]
i	0.00	0.80	0.15	0.00	0.04	0.01
	[0.00, 0.00]	[0.69, 0.90]	[0.07, 0.23]	[0.00, 0.00]	[0.02, 0.07]	[0.00, 0.01]
Δq	0.00	0.05	0.69	0.00	0.26	0.00
	[0.00, 0.00]	[0.05, 0.05]	[0.68, 0.71]	[0.00, 0.00]	[0.25, 0.28]	[0.00, 0.00]
Δs	0.00	0.05	0.64	0.00	0.31	0.00
	[0.00, 0.00]	[0.04, 0.05]	[0.62, 0.66]	[0.00, 0.01]	[0.29, 0.33]	[0.00, 0.00]
q	0.00	0.03	0.35	0.00	0.63	0.00
	[0.00, 0.00]	[0.02, 0.03]	[0.24, 0.48]	[0.00, 0.00]	[0.49, 0.74]	[0.00, 0.00]
s	0.00	0.02	0.28	0.04	0.66	0.00
	[0.00, 0.00]	[0.01, 0.03]	[0.17, 0.39]	[0.00, 0.06]	[0.52, 0.78]	[0.00, 0.00]
$\pi*$	0.00	0.00	0.00	0.00	0.00	1.00
	[0, 0]	[0, 0]	[0, 0]	[0, 0]	[0, 0]	[1, 1]
$y*$	0.00	0.00	0.00	0.00	1.00	0.00
	[0, 0]	[0, 0]	[0, 0]	[0, 0]	[1, 1]	[0, 0]

Note: The square brackets define 95 percent confidence intervals for the contributions from the exogenous shocks

6 Assessing optimal policy

6.1 Method

We take the estimated structural parameters of the model economy as given and minimize the policy maker's loss function by varying the coefficients of the generalized Taylor rule below:

$$i_t = \rho_i i_{t-1} + \tilde{\psi}_\pi \pi_t + \tilde{\psi}_y y_t + \tilde{\psi}_{\Delta e} \Delta y_t + \tilde{\psi}_e \Delta e_t + \varepsilon_{M,t}. \tag{64}$$

Note the omission of the $(1-\rho_i)$ parameter relative to the estimated Taylor rule (41). This allows us to investigate Taylor rules with ρ_i close or equal to one and with non-zero coefficients for the other variables in the rule.

The structural parameters are taken from the posterior means obtained from our empirical estimation. We denote these parameters $\overline{\theta}_s$. The optimal policy parameters are then determined by

$$\theta_p^* = \arg \min_{\theta_p} \mathcal{L}_0(\theta_p|\overline{\theta_s}), \tag{65}$$

subject to the model equations in section 2 and the constraint $0 \leq \rho_i \leq 1$. We solve (65) by running a simplex algorithm in Matlab (fminsearch).

6.2 Optimal policy parameters

[Incomplete]

7 Conclusion

[Incomplete]

References

Adjemian, S., Bastani, H., Juillard, M., Mihoubi, F., Perendia, G., Ratto, M., Villemot, S., 2011. Dynare: Reference manual, version 4. Dynare Working Papers 1, CEPREMAP.
URL http://www.dynare.org

Adolfson, M., Laséen, S., Lindé, J., Villani, M., August 2008. Evaluating an estimated new Keynesian small open economy model. Journal of Economic Dynamics and Control 32 (8), 2690–2721.
URL http://ideas.repec.org/a/eee/dyncon/v32y2008i8p2690-2721.html

Amato, J. D., Laubach, T., 2004. Implications of habit formation for optimal monetary policy. Journal of Monetary Economics 51 (2), 305–325.

Benigno, G., Benigno, P., 2003. Price stability in open economies. The Review of Economic Studies 70 (4), 743–764.

Benigno, P., Woodford, M., December 2005. Inflation stabilization and welfare: The case of a distorted steady state. Journal of the European Economic Association 3 (6), 1185–1236.
URL http://ideas.repec.org/a/tpr/jeurec/v3y2005i6p1185-1236.html

Campa, J. M., Goldberg, L. S., 2005. Exchange rate pass-through into import prices. The Review of Economics and Statistics 87 (4), 679–690.

Chari, V., Kehoe, P., McGrattan, E. R., 2002. Can sticky price models generate volatile and persistent real exchange rates? Review of Economic Studies 69 (3), 533–563.

Chinn, M. D., 2006. The (partial) rehabilitation of interest rate parity in the floating rate era: Longer horizons, alternative expectations, and emerging markets. Journal of International Money and Finance 25, 7–21.

Clarida, R., Galí, J., Gertler, M., Nov. 2001. Optimal monetary policy in closed versus open economies: An integrated approach. NBER Working Papers 8604, National Bureau of Economic Research, Inc.
URL http://ideas.repec.org/p/nbr/nberwo/8604.html

Corsetti, G., Pesenti, P., 2001. Welfare and macroeconomic interdependence. Quarterly Journal of Economics 116 (2), 421–445.
URL http://EconPapers.repec.org/RePEc:tpr:qjecon:v:116:y:2001:i:2:p:421-445

Corsetti, G., Pesenti, P., 2005. International dimensions of optimal monetary policy. Journal of Monetary Economics 52 (2), 281–305.

De Paoli, B., February 2009a. Monetary policy and welfare in a small open economy. Journal of International Economics 77 (1), 11–22.
URL http://ideas.repec.org/a/eee/inecon/v77y2009i1p11-22.html

De Paoli, B., October 2009b. Monetary policy under alternative asset market structures: The case of a small open economy. Journal of Money, Credit and Banking 41 (7), 1301–1330.
URL http://ideas.repec.org/a/mcb/jmoncb/v41y2009i7p1301-1330.html

Engel, C., 1996. The forward discount anomaly and the risk premium: A survey of recent evidence. Journal of Empirical Finance 3, 123–192.

Engel, C., August 2000. Local-currency pricing and the choice of exchange-rate regime. European Economic Review 44 (8), 1449–1472.
URL http://ideas.repec.org/a/eee/eecrev/v44y2000i8p1449-1472.html

Engel, C., November 2009. Exchange rate policies. Staff Papers 8, Federal Reserve Bank of Dallas.
URL http://econpapers.repec.org/RePEc:fip:feddst:y:2009:i:nov:n:8

Faia, E., Monacelli, T., 2008. Optimal monetary policy in a small open economy with home bias. Journal of Money, Credit and Banking 40 (4), 721–750.

Feenstra, R. C., Obstfeld, M., Russ, K. N., 2010. In search of the armington elasticity. University of California-Davis, unpublished mimeo.

Froot, K. A., Thaler, R. H., 1990. Foreign exchange. Journal of Economic Perspectives 4, 179–192.

Galí, J., 2008. Monetary Policy, Inflation, and the Business Cycle: An Introduction to the New Keynesian Framework. Princeton University Press, Princeton.

Galí, J., Monacelli, T., July 2005. Monetary policy and exchange rate volatility in a small open economy. Review of Economic Studies 72 (3), 707–734.

Justiniano, A., Preston, B., 2010a. Can structural small open-economy models account for the influence of foreign variables? Journal of International Economics 81 (1), 61–74.

Justiniano, A., Preston, B., 2010b. Monetary policy and uncertainty in an empirical small open-economy model. Journal of Applied Econometrics 25, 93–128.

Kam, T., Lees, K., Liu, P., 06 2009. Uncovering the hit list for small inflation targeters: A Bayesian structural analysis. Journal of Money, Credit and Banking 41 (4), 583–618.
URL http://ideas.repec.org/a/mcb/jmoncb/v41y2009i4p583-618.html

Kirsanova, T., Leith, C., Wren-Lewis, S., 2006. Should central banks target consumer prices or the exchange rate? The Economic Journal 116, F208–F231.

Liu, P., 2006. A small new keynesian model of the new zealand economy. Reserve Bank of New Zealand Discussion Paper Series DP2006/03, Reserve Bank of New Zealand.
URL http://ideas.repec.org/p/nzb/nzbdps/2006-03.html

Lubik, T., Schorfheide, F., 2006. Monetary policy and exchange rate volatility in a small open economy. Journal of Monetary EconomicsForthcoming.

Lubik, T., Schorfheide, F., 2007. Do central banks respond to exchange rate movements? A structural investigation. Journal of Monetary Economics 54 (4), 1069–1087.

McDaniel, C. A., Balistreri, E. J., Mar. 2003. A discussion on armington trade substitution elasticities. Computational Economics 0303002, EconWPA.

Monacelli, T., 2005. Monetary policy in a low pass-through environment. Journal of Money, Credit and Banking 37 (6), 1047–1066.

Rogoff, K., 1996. The purchasing power parity puzzle. Journal of Economic Literature 34 (2).

Rotemberg, J. J., Woodford, M., 1997. Interest rate rules in an estimated sticky price model. In: Taylor, J. B. (Ed.), Monetary Policy Rules. Vol. 31. University of Chicago Press, Chicago, pp. 57–119, NBER Studies in Business Cycles.

Sarno, L., 2005. Viewpoint: Towards a solution to the puzzles in exchange rate economics: where do we stand? Canadian Journal of Economics 38 (3), 673–708.

Sutherland, A., 2002. A simple second-order solution method for dynamic general-equilibrium models. Discussion Paper 0211, University of St. Andrews.

Svensson, L. E., February 1994. Why exchange rate bands? : Monetary independence in spite of fixed exchange rates. Journal of Monetary Economics 33 (1), 157–199.
URL http://ideas.repec.org/a/eee/moneco/v33y1994i1p157-199.html

Tille, C., 2001. The role of consumption substitutability in the international transmission of monetary shocks. Journal of International Economics 53, 421–444.

Woodford, M., 2003. Interest and Prices: Foundations of a Theory of Monetary Policy. Princeton University Press, Princeton, NJ.

Woodford, M., 2011. Interest and Prices: Foundations of a Theory of Monetary Policy. Princeton University Press.
URL http://books.google.co.nz/books?id=8AlrisNOOpYC

A Log-linear model

This section describes the log-linear model estimated in section 5.

The Euler equation comes from equations (11) and 15:

$$c_t = E_t c_{t+1} - \sigma(i_t - \pi_{t+1}). \tag{66}$$

Taking the goods market clearing condition from equation (44), we can represent domestic output by

$$y_{H,t} = (1-\alpha)c_t + \alpha\eta q_t + \alpha c_t^* + \alpha\eta s_t, \tag{67}$$

where the real exchange rate is given by

$$q_t = \psi_{F,t} + (1-\alpha)s_t \tag{68}$$

Notice that the difference in the terms of trade is given by

$$\Delta s_t = \pi_{F,t} - \pi_{H,t}. \tag{69}$$

The domestic Phillips curve is given by

$$\pi_{H,t} - \delta_H \pi_{H,t-1} = \beta E_t(\pi_{H,t+1} - \delta_H \pi_{H,t}) + \frac{(1-\theta_H)(1-\beta\theta_H)}{\theta_H} mc_t, \tag{70}$$

where from equations (11), (12), (20), and (21) we have

$$mc_t = \alpha s_t + \frac{1}{\varphi} y_{H,t} - (1 + \frac{1}{\varphi})\varepsilon_{a,t} + \frac{1}{\sigma} c_t, \tag{71}$$

which is the real marginal cost of a domestic goods producing firm.

The foreign goods Phillips curve is given by

$$\pi_{F,t} - \delta_F \pi_{F,t-1} = \beta E_t(\pi_{F,t+1} - \delta_F \pi_{F,t}) + \frac{(1-\theta_F)(1-\beta\theta_F)}{\theta_F} \psi_{F,t} + \varepsilon_{cp,t}, \tag{72}$$

where $\varepsilon_{cp,t}$ is a cost push shock.

Domestic CPI can be represented as

$$\pi_t = \pi_{H,t} + \alpha \Delta s_t \tag{73}$$

The international risk sharing equation is given by

$$c_t = y_t^* + \sigma q_t - \sigma \zeta_t \tag{74}$$

Finally, the model is closed with a generalized, monetary policy Taylor rule:

$$i_t = \rho_i i_{t-1} + (1-\rho_i)\left(\psi_\pi \pi_t + \psi_y y_t + \psi_{\Delta_y} \Delta y_t + \psi_{\Delta_e} \Delta e_t\right) + \varepsilon_{M,t}. \tag{75}$$

B Final second order output equation and welfare function

The second order equation for consumption, (50), can be rearranged for the terms of trade and substituted into the second order output equation, (49), everywhere there is a terms of trade term (but is not substituted where $\hat{S}_t^2$). Next take the first order equation for consumption, (48), rearrange it for the terms of trade, and substitute it into all of the multiplicative terms of trade terms (but, again, not where $\hat{S}_t^2$). Now take the first order consumption term, (48), rearrange for the terms of trade, substitute this into the first order output equation, (47), rearrange for $\hat{C}_t^2$, and substitute this into the second order output equation we are manipulating. Finally, rearrange the first order output equation, (47), for the terms of trade, square this and substitute it into the $\hat{S}_t^2$ term. This yields:

$$
\begin{aligned}
\hat{Y}_t = {} & \frac{(\alpha^2\sigma - \alpha^2\eta - 2\alpha\sigma + 2\alpha\eta + \sigma)}{\sigma(1-\alpha)}\hat{C}_t - \frac{\alpha\eta(\alpha-1)(\sigma+\alpha\eta-\alpha\sigma-1)}{(\sigma+2\alpha\eta-2\alpha\sigma-\alpha^2\eta+\alpha^2\sigma)^2}\hat{Y}_t\hat{\zeta}_t \qquad (76)\\
& + \frac{1}{2}\frac{\alpha(\alpha(2\eta\sigma-\eta^2-\sigma^2)(\alpha^2-3\alpha+\eta)-2\eta\sigma+\eta+\sigma^2)}{(\sigma+2\alpha\eta-2\alpha\sigma-\alpha^2\eta+\alpha^2\sigma)^2}(\hat{Y}_t^2 - 2\hat{Y}_t\hat{C}_t^*)\\
& + \left[\frac{\sigma(1+\alpha(\alpha-2)(1-\eta))}{(\alpha^2\sigma-\alpha^2\eta-2\alpha\sigma+2\alpha\eta+\sigma)} - \frac{\eta(\sigma+\alpha\eta-\alpha\sigma-\alpha^2\eta+\alpha^2\sigma)}{(\alpha^2\sigma-\alpha^2\eta-2\alpha\sigma+2\alpha\eta+\sigma)}\right.\\
& \left. - \frac{\alpha\eta(\alpha\eta-\alpha\sigma-\alpha^2\eta+\alpha^2\sigma+1)(1-\alpha)(\sigma-1)}{(\alpha^2\sigma-\alpha^2\eta-2\alpha\sigma+2\alpha\eta+\sigma)^2}\right]\hat{Y}\hat{\psi}_{F,t} + \frac{\eta}{(\alpha-1)}\hat{\psi}_{F,t}\\
& + \frac{1}{2}\left[\frac{\sigma(1+\alpha(\alpha-2)(1-\eta))^2}{(1-\alpha)(\alpha^2\sigma-\alpha^2\eta-2\alpha\sigma+2\alpha\eta+\sigma)} + \frac{2\eta(\sigma+\alpha\eta-\alpha\sigma-\alpha^2\eta+\alpha^2\sigma)(1+\alpha(\alpha-2)(1-\eta))}{(\alpha-1)(\alpha^2\sigma-\alpha^2\eta-2\alpha\sigma+2\alpha\eta+\sigma)}\right.\\
& \left. - \frac{\eta(\eta-\alpha^2\eta+\alpha^2\sigma)}{(\alpha-1)} + \frac{\alpha\eta(\alpha\eta-\alpha\sigma-\alpha^2\eta+\alpha^2\sigma+1)(1-\alpha)^2(\sigma-1)^2}{(\alpha^2\sigma-\alpha^2\eta-2\alpha\sigma+2\alpha\eta+\sigma)^2}\right]\hat{\psi}_{F,t}^2\\
& + \left[\frac{\alpha\eta(\eta-\sigma)}{\sigma} + \frac{\alpha(2\eta-\sigma-\alpha\eta+\alpha\sigma)(1+\alpha(\alpha-2)(1-\eta))}{(1-\alpha)(\alpha^2\sigma-\alpha^2\eta-2\alpha\sigma+2\alpha\eta+\sigma)}\right.\\
& + \frac{\eta\alpha(\sigma+\alpha\eta-\alpha\sigma-\alpha^2\eta+\alpha^2\sigma)(2\eta-\sigma-\alpha\eta+\alpha\sigma)}{\sigma(\alpha-1)(\alpha^2\sigma-\alpha^2\eta-2\alpha\sigma+2\alpha\eta+\sigma)}\\
& \left. + \frac{\alpha\eta(1-\alpha)(\sigma-1)(\alpha\eta-\alpha\sigma-\alpha^2\eta+\alpha^2\sigma+1)}{(\alpha^2\sigma-\alpha^2\eta-2\alpha\sigma+2\alpha\eta+\sigma)^2}\right]\hat{\psi}_{F,t}\hat{C}_t^*\\
& - \frac{\alpha\eta(\alpha-1)\left[\alpha^2(\eta-\sigma)(\eta-1)+\alpha(3\sigma(\eta-1)-(2\eta+1)(\eta-1))+\sigma(2-\eta)-1\right]}{(\sigma+2\alpha\eta-2\alpha\sigma-\alpha^2\eta+\alpha^2\sigma)^2}\hat{\psi}_{F,t}\hat{\zeta}_t + \mathcal{O},
\end{aligned}
$$

which can be substituted into the second order approximation to the utility function.

Assuming an optimal tax:

$$
\frac{Yv'(Y)}{Cu'(C)} = \frac{\sigma(1-\alpha)}{(\alpha^2\sigma-\alpha^2\eta-2\alpha\sigma+2\alpha\eta+\sigma)}, \qquad (77)
$$

we can eliminate the distortion owing to monopolistic power of firms producing domestic goods. Substituting (76) into the second order approximation of the utility function, we have

$$\mathcal{W}_t = \frac{U_t - U}{Cu'(C)} = \frac{1}{2}(1-\frac{1}{\sigma})\widehat{C}_t^2 + \frac{Yv'(Y)}{Cu'(C)}\left\{ -\frac{\eta}{(\alpha-1)}\hat{\psi}_{F,t} \right. \tag{78}$$

$$- \frac{1}{2}\left[\frac{\alpha(\alpha(2\eta\sigma - \eta^2 - \sigma^2)(\alpha^2 - 3\alpha + \eta) - 2\eta\sigma + \eta + \sigma^2)}{(\sigma + 2\alpha\eta - 2\alpha\sigma - \alpha^2\eta + \alpha^2\sigma)^2} + (1+\frac{1}{\psi}) \right] \hat{Y}_t^2$$

$$+ \frac{\alpha\eta(\alpha-1)(\sigma + \alpha\eta - \alpha\sigma - 1)}{(\sigma + 2\alpha\eta - 2\alpha\sigma - \alpha^2\eta + \alpha^2\sigma)^2}\hat{Y}_t\hat{\zeta}_t$$

$$+ \frac{\alpha(\alpha(2\eta\sigma - \eta^2 - \sigma^2)(\alpha^2 - 3\alpha + \eta) - 2\eta\sigma + \eta + \sigma^2)}{(\sigma + 2\alpha\eta - 2\alpha\sigma - \alpha^2\eta + \alpha^2\sigma)^2}\hat{Y}_t\hat{C}_t^*$$

$$- \frac{1}{2}\left[\frac{\sigma(1 + \alpha(\alpha-2)(1-\eta))^2}{(1-\alpha)(\alpha^2\sigma - \alpha^2\eta - 2\alpha\sigma + 2\alpha\eta + \sigma)} \right.$$

$$+ \frac{2\eta(\sigma + \alpha\eta - \alpha\sigma - \alpha^2\eta + \alpha^2\sigma)(1 + \alpha(\alpha-2)(1-\eta))}{(\alpha-1)(\alpha^2\sigma - \alpha^2\eta - 2\alpha\sigma + 2\alpha\eta + \sigma)}$$

$$\left. - \frac{\eta(\eta - \alpha^2\eta + \alpha^2\sigma)}{(\alpha-1)} + \frac{\alpha\eta(\alpha\eta - \alpha\sigma - \alpha^2\eta + \alpha^2\sigma + 1)(1-\alpha)^2(\sigma-1)^2}{(\alpha^2\sigma - \alpha^2\eta - 2\alpha\sigma + 2\alpha\eta + \sigma)^2} \right] \hat{\psi}_{F,t}^2$$

$$- \left[\frac{\sigma(1 + \alpha(\alpha-2)(1-\eta))}{(\alpha^2\sigma - \alpha^2\eta - 2\alpha\sigma + 2\alpha\eta + \sigma)} - \frac{\eta(\sigma + \alpha\eta - \alpha\sigma - \alpha^2\eta + \alpha^2\sigma)}{(\alpha^2\sigma - \alpha^2\eta - 2\alpha\sigma + 2\alpha\eta + \sigma)} \right.$$

$$\left. - \frac{\alpha\eta(\alpha\eta - \alpha\sigma - \alpha^2\eta + \alpha^2\sigma + 1)(1-\alpha)(\sigma-1)}{(\alpha^2\sigma - \alpha^2\eta - 2\alpha\sigma + 2\alpha\eta + \sigma)^2} \right] \hat{Y}\hat{\psi}_{F,t}$$

$$- \left[\frac{\alpha\eta(\eta - \sigma)}{\sigma} + \frac{\alpha(2\eta - \sigma - \alpha\eta + \alpha\sigma)(1 + \alpha(\alpha-2)(1-\eta))}{(1-\alpha)(\alpha^2\sigma - \alpha^2\eta - 2\alpha\sigma + 2\alpha\eta + \sigma)} \right.$$

$$+ \frac{\eta\alpha(\sigma + \alpha\eta - \alpha\sigma - \alpha^2\eta + \alpha^2\sigma)(2\eta - \sigma - \alpha\eta + \alpha\sigma)}{\sigma(\alpha-1)(\alpha^2\sigma - \alpha^2\eta - 2\alpha\sigma + 2\alpha\eta + \sigma)}$$

$$\left. + \frac{\alpha\eta(1-\alpha)(\sigma-1)(\alpha\eta - \alpha\sigma - \alpha^2\eta + \alpha^2\sigma + 1)}{(\alpha^2\sigma - \alpha^2\eta - 2\alpha\sigma + 2\alpha\eta + \sigma)^2} \right] \hat{\psi}_{F,t}\hat{C}_t^*$$

$$+ \frac{\alpha\eta(\alpha-1)\left[\alpha^2(\eta-\sigma)(\eta-1) + \alpha(3\sigma(\eta-1) - (2\eta+1)(\eta-1)) + \sigma(2-\eta) - 1\right]}{(\sigma + 2\alpha\eta - 2\alpha\sigma - \alpha^2\eta + \alpha^2\sigma)^2}\hat{\psi}_{F,t}\hat{\zeta}_t$$

$$\left. -\frac{1}{2}(\frac{1}{\psi} + \frac{1}{\tau})\mathrm{Var}_z\widehat{y}_t(z) \right\} + \text{t.i.p} + \mathcal{O}$$

Now we add and subtract efficient rates, e.g. $\hat{X}_t = (\hat{X}_t - \hat{X}_t^e + \hat{X}_t)$, and expand terms (relegating squared efficient rate variables, and multiplicative efficient rate/shocks to the $t.i.p$ term) giving:

$$
\begin{aligned}
\mathcal{W}_t = {} & \frac{\sigma(1-\alpha)}{(\alpha^2\sigma-\alpha^2\eta-2\alpha\sigma+2\alpha\eta+\sigma)}\Bigg\{-\frac{1}{2}\Bigg[\Bigg(\frac{\alpha(\alpha(2\eta\sigma-\eta^2-\sigma^2)(\alpha^2-3\alpha+\eta)-2\eta\sigma+\eta+\sigma^2)}{(\sigma+2\alpha\eta-2\alpha\sigma-\alpha^2\eta+\alpha^2\sigma)^2} \qquad (79)\\
& +(1+\frac{1}{\psi})\Bigg)-\frac{(\sigma-1)(1-\alpha)}{(\alpha^2\sigma-\alpha^2\eta-2\alpha\sigma+2\alpha\eta+\sigma)}\Bigg]\left(\hat{Y}_t-\hat{Y}_t^e\right)^2-\frac{\eta}{(\alpha-1)}(\hat{\psi}_{F,t}-\hat{\psi}_{F,t})\\
& -\Bigg(\Bigg[\Bigg(\frac{\alpha(\alpha(2\eta\sigma-\eta^2-\sigma^2)(\alpha^2-3\alpha+\eta)-2\eta\sigma+\eta+\sigma^2)}{(\sigma+2\alpha\eta-2\alpha\sigma-\alpha^2\eta+\alpha^2\sigma)^2}+(1+\frac{1}{\psi})\Bigg)\\
& -\frac{(\sigma-1)(1-\alpha)}{(\alpha^2\sigma-\alpha^2\eta-2\alpha\sigma+2\alpha\eta+\sigma)}\Bigg]\frac{\alpha\psi(\eta-\sigma)(\alpha-2)}{\sigma(2\alpha\eta+\sigma+\psi-2\alpha\sigma-\alpha^2\eta+\alpha^2\sigma)}\\
& +\Bigg[\frac{\alpha(\alpha(2\eta\sigma-\eta^2-\sigma^2)(\alpha^2-3\alpha+\eta)-2\eta\sigma+\eta+\sigma^2)}{(\sigma+2\alpha\eta-2\alpha\sigma-\alpha^2\eta+\alpha^2\sigma)^2}\\
& +\frac{(\sigma-1)(\alpha^2\sigma-\alpha^2\eta-\alpha\sigma+2\alpha\eta)}{\sigma(\alpha^2\sigma-\alpha^2\eta-2\alpha\sigma+2\alpha\eta+\sigma)}\Bigg]\Bigg)(\hat{Y}_t-\hat{Y}_t^e)\hat{C}_t^*\\
& -\Bigg(\Bigg[\Bigg(\frac{\alpha(\alpha(2\eta\sigma-\eta^2-\sigma^2)(\alpha^2-3\alpha+\eta)-2\eta\sigma+\eta+\sigma^2)}{(\sigma+2\alpha\eta-2\alpha\sigma-\alpha^2\eta+\alpha^2\sigma)^2}+(1+\frac{1}{\psi})\Bigg)\\
& -\frac{(\sigma-1)(1-\alpha)}{(\alpha^2\sigma-\alpha^2\eta-2\alpha\sigma+2\alpha\eta+\sigma)}\Bigg]\frac{\sigma(1+\varphi)(\alpha^2\sigma-\alpha^2\eta-2\alpha\sigma+2\alpha\eta+\sigma)}{\sigma(2\alpha\eta+\sigma+\varphi-2\alpha\sigma-\alpha^2\eta+\alpha^2\sigma)}+(1+\frac{1}{\psi})\Bigg)(\hat{Y}_t-\hat{Y}_t^e)\hat{\varepsilon}_{a,t}\\
& +\Bigg[\frac{\alpha\eta(\alpha-1)(\sigma+\alpha\eta-\alpha\sigma-1)}{(\sigma+2\alpha\eta-2\alpha\sigma-\alpha^2\eta+\alpha^2\sigma)^2}-\frac{1}{2}\frac{\alpha\eta(\sigma-1)(2-\alpha)}{\sigma(\alpha^2\sigma-\alpha^2\eta-2\alpha\sigma+2\alpha\eta+\sigma)}\Bigg]\left(\hat{Y}_t-\hat{Y}_t^e\right)\hat{\zeta}_t\\
& -\Bigg[\Bigg(\frac{\sigma(1+\alpha(\alpha-2)(1-\eta))}{(\alpha^2\sigma-\alpha^2\eta-2\alpha\sigma+2\alpha\eta+\sigma)}-\frac{\alpha\eta(\alpha\eta-\alpha\sigma-\alpha^2\eta+\alpha^2\sigma+1)(1-\alpha)(\sigma-1)}{(\alpha^2\sigma-\alpha^2\eta-2\alpha\sigma+2\alpha\eta+\sigma)^2}\\
& -\frac{\eta(\sigma+\alpha\eta-\alpha\sigma-\alpha^2\eta+\alpha^2\sigma)}{(\alpha^2\sigma-\alpha^2\eta-2\alpha\sigma+2\alpha\eta+\sigma)}\Bigg)+\frac{(\sigma-1)}{(\alpha^2\sigma-\alpha^2\eta-2\alpha\sigma+2\alpha\eta+\sigma)}\Bigg](\hat{Y}_t-\hat{Y}_t^e)(\hat{\psi}_{F,t}-\hat{\psi}_{F,t})
\end{aligned}
$$

$$
\begin{aligned}
&-\frac{1}{2}\left[\left(\frac{\sigma(1+\alpha(\alpha-2)(1-\eta))^2}{(1-\alpha)(\alpha^2\sigma-\alpha^2\eta-2\alpha\sigma+2\alpha\eta+\sigma)}-\frac{\eta(\eta-\alpha^2\eta+\alpha^2\sigma)}{(\alpha-1)}\right.\right. \qquad (80)\\
&+\frac{2\eta(\sigma+\alpha\eta-\alpha\sigma-\alpha^2\eta+\alpha^2\sigma)(1+\alpha(\alpha-2)(1-\eta))}{(\alpha-1)(\alpha^2\sigma-\alpha^2\eta-2\alpha\sigma+2\alpha\eta+\sigma)}\\
&\left.\left.+\frac{\alpha\eta(\alpha\eta-\alpha\sigma-\alpha^2\eta+\alpha^2\sigma+1)(1-\alpha)^2(\sigma-1)^2}{(\alpha^2\sigma-\alpha^2\eta-2\alpha\sigma+2\alpha\eta+\sigma)^2}\right)-\frac{(\sigma-1)}{(1-\alpha)(\alpha^2\sigma-\alpha^2\eta-2\alpha\sigma+2\alpha\eta+\sigma)}\right]\\
&\times(\hat{\psi}_{F,t}-\hat{\psi}_{F,t})^2\\
&-\left(\left[\left[\left(\frac{\sigma(1+\alpha(\alpha-2)(1-\eta))}{(\alpha^2\sigma-\alpha^2\eta-2\alpha\sigma+2\alpha\eta+\sigma)}-\frac{\eta(\sigma+\alpha\eta-\alpha\sigma-\alpha^2\eta+\alpha^2\sigma)}{(\alpha^2\sigma-\alpha^2\eta-2\alpha\sigma+2\alpha\eta+\sigma)}\right.\right.\right.\right.\\
&\left.\left.-\frac{\alpha\eta(\alpha\eta-\alpha\sigma-\alpha^2\eta+\alpha^2\sigma+1)(1-\alpha)(\sigma-1)}{(\alpha^2\sigma-\alpha^2\eta-2\alpha\sigma+2\alpha\eta+\sigma)^2}\right)+\frac{(\sigma-1)}{(\alpha^2\sigma-\alpha^2\eta-2\alpha\sigma+2\alpha\eta+\sigma)}\right]\times\\
&\frac{\alpha\psi(\eta-\sigma)(\alpha-2)}{\sigma(2\alpha\eta+\sigma+\psi-2\alpha\sigma-\alpha^2\eta+\alpha^2\sigma)}-\left[\left(\frac{\alpha\eta(\eta-\sigma)}{\sigma}+\frac{\alpha(2\eta-\sigma-\alpha\eta+\alpha\sigma)(1+\alpha(\alpha-2)(1-\eta))}{(1-\alpha)(\alpha^2\sigma-\alpha^2\eta-2\alpha\sigma+2\alpha\eta+\sigma)}\right.\right.\\
&\left.+\frac{\eta\alpha(\sigma+\alpha\eta-\alpha\sigma-\alpha^2\eta+\alpha^2\sigma)(2\eta-\sigma-\alpha\eta+\alpha\sigma)}{\sigma(\alpha-1)(\alpha^2\sigma-\alpha^2\eta-2\alpha\sigma+2\alpha\eta+\sigma)}+\frac{\alpha\eta(1-\alpha)(\sigma-1)(\alpha\eta-\alpha\sigma-\alpha^2\eta+\alpha^2\sigma+1)}{(\alpha^2\sigma-\alpha^2\eta-2\alpha\sigma+2\alpha\eta+\sigma)^2}\right)\\
&\left.\left.+\frac{(\sigma-1)(\alpha^2\sigma-\alpha^2\eta-\alpha\sigma+2\alpha\eta)}{\sigma(1-\alpha)(\alpha^2\sigma-\alpha^2\eta-2\alpha\sigma+2\alpha\eta+\sigma)}\right]\right)(\hat{\psi}_{F,t}-\hat{\psi}_{F,t})\hat{C}_t^*\\
&-\left[\left(\frac{\sigma(1+\alpha(\alpha-2)(1-\eta))}{(\alpha^2\sigma-\alpha^2\eta-2\alpha\sigma+2\alpha\eta+\sigma)}-\frac{\eta(\sigma+\alpha\eta-\alpha\sigma-\alpha^2\eta+\alpha^2\sigma)}{(\alpha^2\sigma-\alpha^2\eta-2\alpha\sigma+2\alpha\eta+\sigma)}\right.\right.\\
&\left.\left.-\frac{\alpha\eta(\alpha\eta-\alpha\sigma-\alpha^2\eta+\alpha^2\sigma+1)(1-\alpha)(\sigma-1)}{(\alpha^2\sigma-\alpha^2\eta-2\alpha\sigma+2\alpha\eta+\sigma)^2}\right)+\frac{(\sigma-1)}{(\alpha^2\sigma-\alpha^2\eta-2\alpha\sigma+2\alpha\eta+\sigma)}\right] \qquad (81)\\
&\times\frac{\sigma(1+\varphi)(\alpha^2\sigma-\alpha^2\eta-2\alpha\sigma+2\alpha\eta+\sigma)}{\sigma(2\alpha\eta+\sigma+\varphi-2\alpha\sigma-\alpha^2\eta+\alpha^2\sigma)}(\hat{\psi}_{F,t}-\hat{\psi}_{F,t})\varepsilon_{a,t}\\
&+\left[\frac{\alpha\eta(\alpha-1)\left[\alpha^2(\eta-\sigma)(\eta-1)+\alpha(3\sigma(\eta-1)-(2\eta+1)(\eta-1))+\sigma(2-\eta)-1\right]}{(\sigma+2\alpha\eta-2\alpha\sigma-\alpha^2\eta+\alpha^2\sigma)^2}\right.\\
&\left.\left.+\frac{1}{2}\frac{\alpha\eta(\sigma-1)(2-\alpha)}{\sigma(1-\alpha)(\alpha^2\sigma-\alpha^2\eta-2\alpha\sigma+2\alpha\eta+\sigma)}\right](\hat{\psi}_{F,t}-\hat{\psi}_{F,t})\hat{\zeta}_t-\frac{1}{2}\tau\mathrm{Var}_j\widehat{p}_{H,t}(j)\right\}+\mathrm{t.i.p}+\mathcal{O} \qquad (82)
\end{aligned}
$$

As derived in Woodford (2011) and Galí (2008), inflation enters via the term:

$$
\sum_{t=0}^{\infty}\beta^t\mathrm{Var}_i\log y_t(i)=\frac{\theta_H\tau^2}{(1-\theta_H)(1-\beta\theta_H)}\sum_{t=0}^{\infty}\beta^t(\pi_t-\delta_H\pi_{t-1})^2+\mathrm{t.i.p.} \qquad (83)
$$

See Woodford (2011) for a derivation.

In the text we summarize the welfare function as:

$$
\begin{aligned}
\mathcal{W}_t=&A_1\left(\hat{Y}_t-\hat{Y}_t^e\right)^2+A_2(\hat{\psi}_{F,t}-\hat{\psi}_{F,t})+A_3(\hat{Y}_t-\hat{Y}_t^e)\hat{Y}_t^* \qquad (84)\\
&+A_4(\hat{Y}_t-\hat{Y}_t^e)\hat{\varepsilon}_{a,t}+A_5\left(\hat{Y}_t-\hat{Y}_t^e\right)\hat{\zeta}_t+A_6(\hat{Y}_t-\hat{Y}_t^e)(\hat{\psi}_{F,t}-\hat{\psi}_{F,t})\\
&+A_7(\hat{\psi}_{F,t}-\hat{\psi}_{F,t})^2+A_8(\hat{\psi}_{F,t}-\hat{\psi}_{F,t})\hat{Y}_t^*+A_9(\hat{\psi}_{F,t}-\hat{\psi}_{F,t})\hat{\varepsilon}_{a,t}\\
&+A_{10}(\hat{\psi}_{F,t}-\hat{\psi}_{F,t})\hat{\zeta}_t+A_{11}\mathrm{Var}_j\widehat{p}_{H,t}(j)+\mathrm{t.i.p}+\mathcal{O}
\end{aligned}
$$

where the coefficients are given by the structural parameters of the model:

$$A_1 = -\frac{Yv_y}{CU_c}\frac{1}{2}\left[\left(\frac{\alpha(\alpha(2\eta\sigma-\eta^2-\sigma^2)(\alpha^2-3\alpha+\eta)-2\eta\sigma+\eta+\sigma^2)}{(\sigma+2\alpha\eta-2\alpha\sigma-\alpha^2\eta+\alpha^2\sigma)^2}+(1+\frac{1}{\psi})\right)\right. \tag{85}$$
$$\left.-\frac{(\sigma-1)(1-\alpha)}{(\alpha^2\sigma-\alpha^2\eta-2\alpha\sigma+2\alpha\eta+\sigma)}\right],$$
$$A_2 = -\frac{Yv_y}{CU_c}\frac{\eta}{(\alpha-1)},$$
$$A_3 = -\frac{Yv_y}{CU_c}\left(\left[\left(\frac{\alpha(\alpha(2\eta\sigma-\eta^2-\sigma^2)(\alpha^2-3\alpha+\eta)-2\eta\sigma+\eta+\sigma^2)}{(\sigma+2\alpha\eta-2\alpha\sigma-\alpha^2\eta+\alpha^2\sigma)^2}+(1+\frac{1}{\psi})\right)\right.\right.$$
$$\left.-\frac{(\sigma-1)(1-\alpha)}{(\alpha^2\sigma-\alpha^2\eta-2\alpha\sigma+2\alpha\eta+\sigma)}\right]\frac{\alpha\psi(\eta-\sigma)(\alpha-2)}{\sigma(2\alpha\eta+\sigma+\psi-2\alpha\sigma-\alpha^2\eta+\alpha^2\sigma)}$$
$$\left.-\frac{\alpha(\alpha(2\eta\sigma-\eta^2-\sigma^2)(\alpha^2-3\alpha+\eta)-2\eta\sigma+\eta+\sigma^2)}{(\sigma+2\alpha\eta-2\alpha\sigma-\alpha^2\eta+\alpha^2\sigma)^2}-\frac{(\sigma-1)(\alpha^2\sigma-\alpha^2\eta-\alpha\sigma+2\alpha\eta)}{\sigma(\alpha^2\sigma-\alpha^2\eta-2\alpha\sigma+2\alpha\eta+\sigma)}\right),$$
$$A_4 = -\frac{Yv_y}{CU_c}\left(\left[\left(\frac{\alpha(\alpha(2\eta\sigma-\eta^2-\sigma^2)(\alpha^2-3\alpha+\eta)-2\eta\sigma+\eta+\sigma^2)}{(\sigma+2\alpha\eta-2\alpha\sigma-\alpha^2\eta+\alpha^2\sigma)^2}+(1+\frac{1}{\psi})\right)\right.\right.$$
$$\left.\left.-\frac{(\sigma-1)(1-\alpha)}{(\alpha^2\sigma-\alpha^2\eta-2\alpha\sigma+2\alpha\eta+\sigma)}\right]\frac{\sigma(1+\varphi)(\alpha^2\sigma-\alpha^2\eta-2\alpha\sigma+2\alpha\eta+\sigma)}{\sigma(2\alpha\eta+\sigma+\varphi-2\alpha\sigma-\alpha^2\eta+\alpha^2\sigma)}-(1+\frac{1}{\psi})\right),$$
$$A_5 = \frac{Yv_y}{CU_c}\left[\frac{\alpha\eta(\alpha-1)(\sigma+\alpha\eta-\alpha\sigma-1)}{(\sigma+2\alpha\eta-2\alpha\sigma-\alpha^2\eta+\alpha^2\sigma)^2}-\frac{1}{2}\frac{\alpha\eta(\sigma-1)(2-\alpha)}{\sigma(\alpha^2\sigma-\alpha^2\eta-2\alpha\sigma+2\alpha\eta+\sigma)}\right],$$
$$A_6 = -\frac{Yv_y}{CU_c}\left[\left(\frac{\sigma(1+\alpha(\alpha-2)(1-\eta))}{(\alpha^2\sigma-\alpha^2\eta-2\alpha\sigma+2\alpha\eta+\sigma)}-\frac{\alpha\eta(\alpha\eta-\alpha\sigma-\alpha^2\eta+\alpha^2\sigma+1)(1-\alpha)(\sigma-1)}{(\alpha^2\sigma-\alpha^2\eta-2\alpha\sigma+2\alpha\eta+\sigma)^2}\right.\right.$$
$$\left.\left.-\frac{\eta(\sigma+\alpha\eta-\alpha\sigma-\alpha^2\eta+\alpha^2\sigma)}{(\alpha^2\sigma-\alpha^2\eta-2\alpha\sigma+2\alpha\eta+\sigma)}\right)+\frac{(\sigma-1)}{(\alpha^2\sigma-\alpha^2\eta-2\alpha\sigma+2\alpha\eta+\sigma)}\right],$$
$$A_7 = -\frac{Yv_y}{CU_c}\frac{1}{2}\left[\left(\frac{\sigma(1+\alpha(\alpha-2)(1-\eta))^2}{(1-\alpha)(\alpha^2\sigma-\alpha^2\eta-2\alpha\sigma+2\alpha\eta+\sigma)}\right.\right.$$
$$+\frac{2\eta(\sigma+\alpha\eta-\alpha\sigma-\alpha^2\eta+\alpha^2\sigma)(1+\alpha(\alpha-2)(1-\eta))}{(\alpha-1)(\alpha^2\sigma-\alpha^2\eta-2\alpha\sigma+2\alpha\eta+\sigma)}$$
$$\left.-\frac{(\sigma-1)}{(1-\alpha)(\alpha^2\sigma-\alpha^2\eta-2\alpha\sigma+2\alpha\eta+\sigma)}\right],$$
$$A_8 = -\frac{Yv_y}{CU_c}\left(\left[\frac{\sigma(1+\alpha(\alpha-2)(1-\eta))}{(\alpha^2\sigma-\alpha^2\eta-2\alpha\sigma+2\alpha\eta+\sigma)}-\frac{\eta(\sigma+\alpha\eta-\alpha\sigma-\alpha^2\eta+\alpha^2\sigma)}{(\alpha^2\sigma-\alpha^2\eta-2\alpha\sigma+2\alpha\eta+\sigma)}\right.\right.$$
$$\left.-\frac{\alpha\eta(\alpha\eta-\alpha\sigma-\alpha^2\eta+\alpha^2\sigma+1)(1-\alpha)(\sigma-1)}{(\alpha^2\sigma-\alpha^2\eta-2\alpha\sigma+2\alpha\eta+\sigma)^2}+\frac{(\sigma-1)}{(\alpha^2\sigma-\alpha^2\eta-2\alpha\sigma+2\alpha\eta+\sigma)}\right]\times$$
$$\frac{\alpha\psi(\eta-\sigma)(\alpha-2)}{\sigma(2\alpha\eta+\sigma+\psi-2\alpha\sigma-\alpha^2\eta+\alpha^2\sigma)}+\frac{\alpha\eta(\eta-\sigma)}{\sigma}+\frac{\alpha(2\eta-\sigma-\alpha\eta+\alpha\sigma)(1+\alpha(\alpha-2)(1-\eta))}{(1-\alpha)(\alpha^2\sigma-\alpha^2\eta-2\alpha\sigma+2\alpha\eta+\sigma)}$$
$$+\frac{\eta\alpha(\sigma+\alpha\eta-\alpha\sigma-\alpha^2\eta+\alpha^2\sigma)(2\eta-\sigma-\alpha\eta+\alpha\sigma)}{\sigma(\alpha-1)(\alpha^2\sigma-\alpha^2\eta-2\alpha\sigma+2\alpha\eta+\sigma)}+\frac{\alpha\eta(1-\alpha)(\sigma-1)(\alpha\eta-\alpha\sigma-\alpha^2\eta+\alpha^2\sigma+1)}{(\alpha^2\sigma-\alpha^2\eta-2\alpha\sigma+2\alpha\eta+\sigma)^2}\Bigg) \tag{86}$$
$$\left.+\frac{(\sigma-1)(\alpha^2\sigma-\alpha^2\eta-\alpha\sigma+2\alpha\eta)}{\sigma(1-\alpha)(\alpha^2\sigma-\alpha^2\eta-2\alpha\sigma+2\alpha\eta+\sigma)}\right)$$

$$
\begin{aligned}
A_9 =& -\frac{Yv_y}{CU_c}\left[\frac{\sigma(1+\alpha(\alpha-2)(1-\eta))}{(\alpha^2\sigma-\alpha^2\eta-2\alpha\sigma+2\alpha\eta+\sigma)}-\frac{\eta(\sigma+\alpha\eta-\alpha\sigma-\alpha^2\eta+\alpha^2\sigma)}{(\alpha^2\sigma-\alpha^2\eta-2\alpha\sigma+2\alpha\eta+\sigma)}\right.\\
&\left.-\frac{\alpha\eta(\alpha\eta-\alpha\sigma-\alpha^2\eta+\alpha^2\sigma+1)(1-\alpha)(\sigma-1)}{(\alpha^2\sigma-\alpha^2\eta-2\alpha\sigma+2\alpha\eta+\sigma)^2}+\frac{(\sigma-1)}{(\alpha^2\sigma-\alpha^2\eta-2\alpha\sigma+2\alpha\eta+\sigma)}\right]\\
&\times\frac{\sigma(1+\varphi)(\alpha^2\sigma-\alpha^2\eta-2\alpha\sigma+2\alpha\eta+\sigma)}{\sigma(2\alpha\eta+\sigma+\varphi-2\alpha\sigma-\alpha^2\eta+\alpha^2\sigma)}\\
A_{10} =& \frac{Yv_y}{CU_c}\left[\frac{\alpha\eta(\alpha-1)\left[\alpha^2(\eta-\sigma)(\eta-1)+\alpha(3\sigma(\eta-1)-(2\eta+1)(\eta-1))+\sigma(2-\eta)-1\right]}{(\sigma+2\alpha\eta-2\alpha\sigma-\alpha^2\eta+\alpha^2\sigma)^2}\right.\\
&\left.+\frac{1}{2}\frac{\alpha\eta(\sigma-1)(2-\alpha)}{\sigma(1-\alpha)(\alpha^2\sigma-\alpha^2\eta-2\alpha\sigma+2\alpha\eta+\sigma)}\right]\\
A_{11} =& -\frac{Yv_y}{CU_c}\frac{1}{2}\frac{\theta_H\tau}{(1-\theta_H)(1-\beta\theta_H)}
\end{aligned}
\tag{87}
$$

where we have

$$
\frac{Yv_y}{CU_c}=\frac{\sigma(1-\alpha)}{(\alpha^2\sigma-\alpha^2\eta-2\alpha\sigma+2\alpha\eta+\sigma)}
\tag{88}
$$